Graphing the curve $y = \frac{xe^{-x}}{x+\frac{1}{2}}$

Let $f(x) = \frac{xe^{-x}}{x+\frac{1}{2}}$. We wish to create a sketch of the graph of f.

We begin by noting the domain of f. We see that the numerator is defined for all x, and the denominator is defined for all x. The denominator is non-zero as long as $x \neq -\frac{1}{2}$. We conclude that the domain is all $x \neq -\frac{1}{2}$.

Thus, we have the potential of a vertical asymptote at $x = -\frac{1}{2}$. We evaluate limits to determine if this is in actuality an asymptote:

$$\lim_{x \to -\frac{1}{2}^+} \frac{xe^{-x}}{x + \frac{1}{2}} = -\infty$$

since the denominator approaches zero while staying positive and the numerator has a non-zero, negative, limit;

$$\lim_{x \to -\frac{1}{2}^-} \frac{xe^{-x}}{x + \frac{1}{2}} = \infty$$

since the denominator approaches zero while staying negative and the numerator has a non-zero, negative, limit.

Either of these limits, by itself, is enough for us to conclude a vertical asymptote at $x = -\frac{1}{2}$, but it is helpful to have both to get a fuller picture.

To complete the analysis of the function's asymptotes, we investigate the function as x approaches ∞ and $-\infty$.

To consider x approaching positive infinity, it is helpful to write

$$f(x) = \frac{xe^{-x}}{x + \frac{1}{2}} = \frac{x}{e^x(x + \frac{1}{2})}.$$

By writing it this way, we have an "infinity over infinity" form as x approaches ∞, and so L'Hospital's rule applies:

$$\lim_{x \to \infty} \frac{x}{e^x(x + \frac{1}{2})} = \lim_{x \to \infty} \frac{1}{e^x(x + \frac{1}{2}) + e^x} = 0$$

since the denominator approaches infinity while the numerator is constant.

Thus, the graph will approach the positive x-axis as x heads toward infinity.

On the other hand,

$$\lim_{x \to -\infty} f(x) = \lim_{x \to \infty} \frac{xe^{-x}}{x + \frac{1}{2}}$$

has an "infinity over infinity" form, since e^{-x} grows without bound as x approaches $-\infty$. Hence, L'Hospital's rule again applies, and we have

$$\lim_{x \to \infty} \frac{xe^{-x}}{x + \frac{1}{2}} = \lim_{x \to \infty} \frac{e^{-x} - xe^{-x}}{1} = \infty.$$

Thus, there is no horizontal asymptote approached as x approaches $-\infty$.

We now investigate where the function is increasing and where it is decreasing.

Differentiating $f(x)$, we have

$$f'(x) = \frac{(e^{-x} - xe^{-x})(x + \frac{1}{2}) - xe^{-x}}{(x + \frac{1}{2})^2} = \frac{e^{-x}(-x^2 - \frac{1}{2}x + \frac{1}{2})}{(x + \frac{1}{2})^2} = \frac{-e^{-x}((x + \frac{1}{4})^2 - \frac{9}{16})}{(x + \frac{1}{2})^2}$$

and so we see that $f'(x) = 0$ only when $x = -\frac{1}{4} \pm \frac{3}{4}$, i.e., at $x = -1$ and $x = \frac{1}{2}$.

This gives us four intervals to consider: (a) $x \leq -1$ (b) $-1 \leq x \leq -\frac{1}{2}$ (c) $-\frac{1}{2} \leq x \leq \frac{1}{2}$ and (d) $x \geq \frac{1}{2}$. We can evaluate $f'(x)$ in each interval to determine whether f is increasing or decreasing there.

Choosing convenient points in each interval we find

$$f'(-2) = -8.21... < 0$$

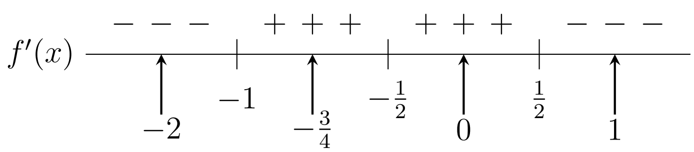
$$f'(-3/4) = 10.585... > 0$$

$$f'(0) = 2 > 0$$

$$f'(1) = -0.1635... < 0$$

Thus, f is increasing on $-1 \leq x \leq -\frac{1}{2}$ and on $-\frac{1}{2} \leq x \leq \frac{1}{2}$, and so decreasing on $x \leq -1$ and $x \geq \frac{1}{2}$.

We can diagram this situation like this:



From this we may conclude that $f(x)$ has a **local minimum** at $x = -1$ and a **local maximum** at $x = \frac{1}{2}$.

We now consider the concavity of f.

The second derivative of f is a bit messy, but we can (eventually) simplify it to

$$f''(x) = \frac{e^{-x}(x^3 + x^2 - \frac{3}{4}x - \frac{3}{2})}{(x + \frac{1}{2})^4}$$

We know e^{-x} is always positive, as is $(x + \frac{1}{2})^4$, so the only thing determining the sign of $f''(x)$ is the cubic expression

$$x^3 + x^2 - \frac{3}{4}x - \frac{3}{2}.$$

Cubic function always have at least one root. If we let $g(x) = x^3 + x^2 - \frac{3}{4}x - \frac{3}{2}$, we see, in fact that $g(0) = -\frac{3}{2} < 0$ and $g(2) = 9$. By the Intermediate Value Theorem, we can conclude that $g(x)$ equals zero for an x between 0 and 2. Further, $g'(x) = 3x^2 + 2x - \frac{3}{4} = 0$ at

$$x = \frac{-2 \pm \sqrt{13}}{6}$$

and $g'(x)$ is negative between these two values. Hence, $g(x)$ has its only local maximum at (-0.9343,-0.7419), and hence $g(x)$ has only one root.

Thus $f''(x)$ changes sign at only one x value, and we can use **Newton's method** to approximate it. We find the only inflection point of f occurs at $x = 1.0558471104...$.

Checking the value of f'', we find that f is concave up for $x \leq -\frac{1}{2}$, concave down for $-\frac{1}{2} \leq x \leq 1.0558471104...$ and concave up for $x \geq 1.0558471104...$.

Now, to sketch the curve, we collect all our features of interest (local extrema, inflection points, asymptotes), finding y-coordinates for them if we have not already done so:

- Local minimum at $(-1, 5.437...)$
- Local maximum at $(\frac{1}{2}, 0.303...)$
- Inflection point at $(1.06..., 0.236...)$
- Vertical asymptote $x = -\frac{1}{2}$
- Horizontal asymptote of $y = 0$ as $x \to \infty$

Notice that we don't need a huge amount of precision on the coordinates to sketch the graph.