Linear Approximation and Newton's Method Worksheet

A very famous and powerful application of the tangent line approximation idea is **Newton's Method** for finding approximations of roots of equations. Say we want to find a solution to an equation

$$f(x) = 0.$$

So, we want a value, r, such that $f(r) = 0$. If the function f is not of a rather particular type, such as linear or quadratic, we generally would have a hard time finding r. In such cases, we often resort to finding an approximation of r using Newton's Method, which is based on the following idea.

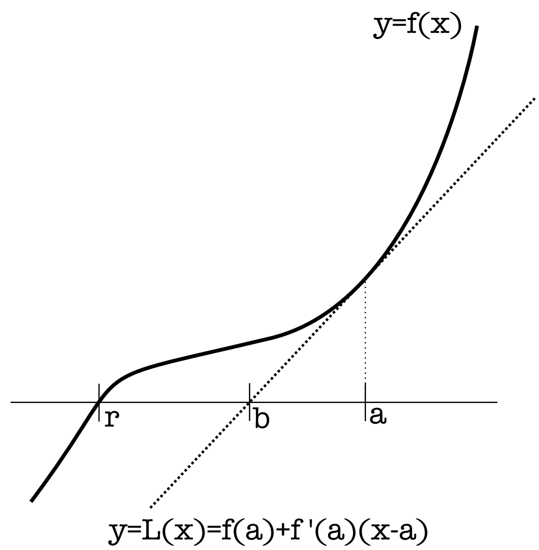
If we are looking for a root r, we might start with a value $x = a$ as an estimate of r. We then improve the estimate by using the **linear approximation** of $f(x)$ at a, and finding the root of the linear approximation. This gives us a new approximation b, which, in many cases will be a better estimate than a.

The linear approximation of $f(x)$ at $x = a$ is

$$L(x) = f(a) + f'(a)(x - a),$$

and if we set this equal to zero and solve for x we find

$$b = a - \frac{f(a)}{f'(a)}$$



The real power of the method, though, comes from the idea that if b is a better estimate for r, we can repeat the method starting at b, and get a new, even better estimate, and we can keep repeating this process as long as we want.

Here's the formulation of the method.

1. Start with an estimate (i.e., a guess) of r. Let's call that guess r_1.
2. Create the **recursive formula**:
$$r_{n+1} = r_n - \frac{f(r_n)}{f'(r_n)}$$
3. Use the formula repeatedly, to generate $r_2, r_3, r_4, \ldots$, until the values you get don't change much (or you come to the conclusion that this method is not working).

For instance, suppose we want a root of the equation $x^2 - 2 = 0$. We could solve this algebraically, but for the sake of example, let's see what Newton's Method does with it.

Say we start with the guess $r_1 = 1.5$. Our recursive formula is

$$r_{n+1} = r_n - \frac{r_n^2 - 2}{2r_n}$$

Plugging in $r_1 = 1.5$ gives us

$$r_2 = 1.5 - \frac{1.5^2 - 2}{2(1.5)} = 1.41666666666666666666666666.$$

Plugging that into the formula, and repeating, gives us the sequence

$$r_3 = 1.41421568627450980392156862 7$$

$$r_4 = 1.41421356237468991062629557 8$$

$$r_5 = 1.41421356237309504880168962 3$$

$$r_6 = 1.41421356237309504880168872 4$$

$$r_7 = 1.41421356237309504880168872 4$$

Since r_6 and r_7 are equal, every additional application of the formula will give the same result, so this is our best approximation of the root of the equation $x^2 - 2 = 0$ that we can get with this method. (This all depends as well on the accuracy of our calculating device: if your calculator presents fewer digits, you might have seen no change earlier in the sequence).

Examples

1. Use Newton's method to find a solution to $x^2 - 17 = 0$.
2. (a) Show that when applying Newton's method to equations of the form $x^2 - B = 0$, the result can be simplified to
$$r_{n+1} = \frac{1}{2}\left(r_n + \frac{B}{r_n}\right)$$
   (b) Use the simplified formula to find the square root of 23.
   (c) What effect does using different starting guesses have?
3. Find the solution to $\cos x = x$ (make a sketch to help you make a first guess).
4. Use Newton's method to find $\ln 2$ (hint: start by finding an equation whose solution is $\ln 2$). What's a reasonable initial guess? What happens if you start with an initial guess of -4? What's going on?
5. The equation $x^2 = 2^x$ has two integer solutions: $x = 2$ and $x = 4$. Use Newton's method to approximate the other solution.