Math 125 D Autumn 2023
Mid-Term Exam Number One
October 19, 2023
Solutions

1. Evaluate the following indefinite integrals.

(a) $\displaystyle\int \left(\sqrt{x}+1\right)\left(\sqrt{x}+2\right)\,dx$

$$\begin{aligned}\int \left(\sqrt{x}+1\right)\left(\sqrt{x}+2\right)\,dx &= \int (x+3\sqrt{x}+2)\,dx \\ &= \frac{1}{2}x^2+2x^{3/2}+2x+C.\end{aligned}$$

(b) $\displaystyle\int \frac{x^3+3x^2+1}{x^2}\,dx$

$$\begin{aligned}\int \frac{x^3+3x^2+1}{x^2}\,dx &= \int \left(x+3+x^{-2}\right)\,dx \\ &= \frac{1}{2}x^2+3x-\frac{1}{x}+C.\end{aligned}$$

(c) $\displaystyle\int x^5\sqrt{x^3+1}\,dx$

Let $u = x^3+1$ so that $du = 3x^2\,dx$.

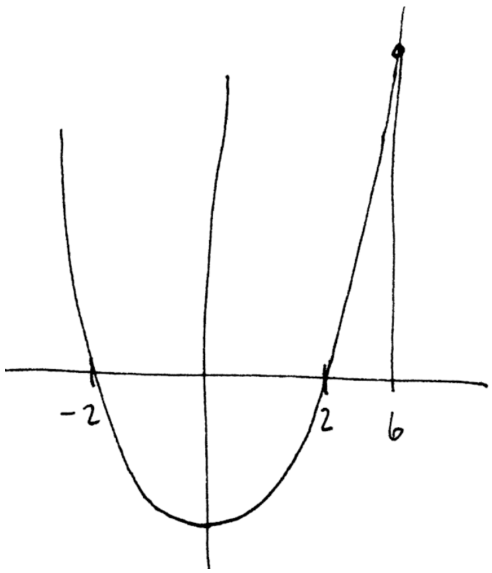
Then $x^5 = x^3 \cdot x^2 = (u-1)\dfrac{du}{3}$.

Thus the integral becomes

$$\frac{1}{3}\int (u-1)\sqrt{u}\,du = \frac{1}{3}\int (u^{3/2}-u^{1/2})\,du = \frac{2}{15}u^{5/2}-\frac{2}{9}u^{3/2}+C = \frac{2}{15}(x^3+1)^{5/2}-\frac{2}{9}(x^3+1)^{3/2}+C.$$

2. Evaluate the following definite integrals.

(a) $\int_0^6 |x^2 - 4|\, dx.$



$x^2 - 4 = 0$

$x^2 = 4$

$x = \pm 2$

By sketching $y = x^2 - 4$ and finding its root, we see that $x^2 - 4 \geq 0$ when $x \geq 2$ and $x^2 - 4 < 0$ when $-2 \leq x \leq 2$, so that

$$\int_0^6 |x^2 - 4|\, dx = \int_0^2 (4 - x^2)\, dx + \int_2^6 (x^2 - 4)\, dx = (4x - \frac{1}{3}x^3)\Big|_0^2 + (\frac{1}{3}x^2 - 4x)\Big|_2^6 = \frac{176}{3}.$$

(b) $\int_{-4}^{4} f(t)\, dt$ where $f(t) = g'(t)$ and $g(t) = te^{2t}$.

Since $f(t) = g'(t)$, $\int f(t)\, dt = g(t) + C$, and so

$$\int_{-4}^{4} f(t)\, dt = g(4) - g(-4) = 4e^8 - (-4e^{-8}) = 4e^8 + 4e^{-8}.$$

(c) $\int_{-1}^{1} \frac{e^x}{e^x + 1}\, dx$

Let $u = e^x + 1$. Then $du = e^x\, dx$. Then

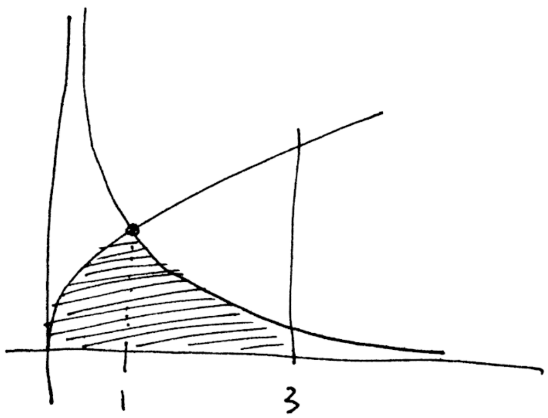
$$\int \frac{e^x}{e^x + 1}\, dx = \int \frac{du}{u} = \ln|u| + C = \ln|e^x + 1| + C.$$

Hence,

$$\int_{-1}^{1} \frac{e^x}{e^x + 1}\, dx = \ln|e^x + 1|\Big|_{-1}^{1}$$

$$= \ln|e + 1| - \ln|e^{-1} + 1| = \ln|e + 1| - \ln|\frac{1 + e}{e}| = \ln|e + 1| - \ln|1 + e| - \ln e = 1.$$

3. Find the area of the region bounded by the curves $y = \dfrac{1}{x^2}$, $y = \sqrt{x}$, $x = 3$ and the x-axis.



$$\frac{1}{x^2} = \sqrt{x}$$

$$\frac{1}{x^4} = x$$

$$1 = x^5$$

$$x = 1$$

After drawing a good sketch, we find that the two curves intersect at $x = 1$ with the region being bounded above by $y = \sqrt{x}$ and below by $y = 0$ for $0 \le x \le 1$, and the region being bounded above by $y = \dfrac{1}{x^2}$ and below by $y = 0$ for $1 \le x \le 3$.

From this, we conclude that the area is

$$\int_0^1 \sqrt{x}\,dx + \int_1^3 \frac{1}{x^2}\,dx = \frac{2}{3}x^{3/2}\bigg|_0^1 + \left(-\frac{1}{x}\right)\bigg|_1^3$$
$$= \frac{4}{3}.$$

4. You find yourself on a distant planet, where the acceleration due to gravity is not the same as on Earth.

   To measure the acceleration due to gravity, you perform an experiment.

   You construct a 50 meter tall tower. From the top of the tower, you throw a rock downward.

   The rock hits the ground exactly 6 seconds later.

   The final 10 meters of its fall takes exactly 1 second.

   What is the acceleration due to gravity on this distant planet?

   We begin with the assumption that acceleration is constant; let g be the acceleration due to gravity on this planet.

   Let $h(t)$ be height of the rock t seconds after it is thrown.

   Then
$$h''(t) = g$$
   so that
$$h'(t) = gt + C_1$$
   and
$$h(t) = \frac{1}{2}gt^2 + C_1 t + C_2$$
   for some constants C_1 and C_2, by integration.

   We know three things:

   - $h(0) = 50$
   - $h(6) = 0$
   - $h(5) = 10$ since the rock travels 10 meters in the 1 second before it reaches the ground.

   Hence,
$$h(0) = 50 = 0 + 0 + C_2$$
   so $C_2 = 50$, and
$$0 = 18g + 6C_1 + 50 \text{ and } 10 = 12.5g + 5C_1 + 50$$
   Using these last two equations and solving for g we find
$$g = -\frac{2}{3}$$
   so the acceleration due to gravity on this planet is $\frac{2}{3}$ m/s^2.

5. Let $g(x) = \cos x \int_{3x}^{x^3} e^{t^2}\,dt$.

Find $g'(x)$ (your answer may involve an integral or two).

By the Fundamental Theorem of Calculus, Part I, we know that if

$$f(x) = \int_a^x h(t)\,dt,$$

then $f'(x) = h(x)$. Hence, if

$$j(u) = \int_a^u h(t)\,dt,$$

and u is a function of x, then

$$\frac{d}{du} j(u)\,dt = h(u)$$

and, by the chain rule,

$$\frac{d}{dx} j(u) = \frac{d}{du} j(u) \cdot \frac{du}{dx}.$$

Thus,

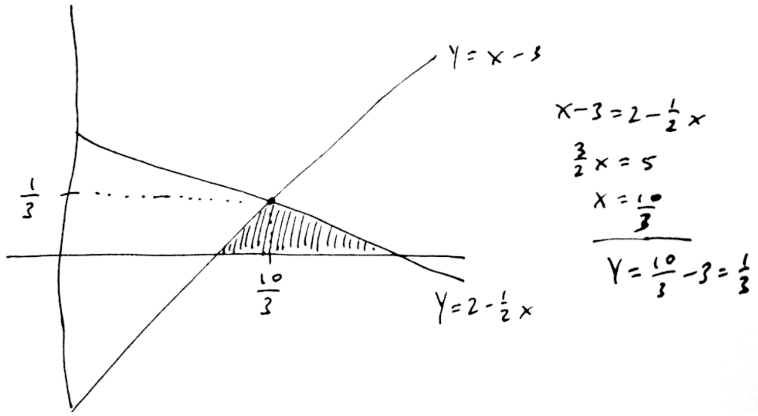
$$\begin{aligned}\frac{d}{dx}\int_{3x}^{x^3} e^{t^2}\,dt &= \frac{d}{dx}\int_0^{x^3} e^{t^2}\,dt - \frac{d}{dx}\int_0^{3x}\,dt\\ &= e^{(x^3)^2}\frac{d}{dx}x^3 - e^{(3x)^2}\frac{d}{dx}3x\\ &= 3x^2e^{x^6} - 3e^{9x^2}\end{aligned}$$

Hence, using the product rule, we find

$$g'(x) = -\sin x \int_{3x}^{x^3} e^{t^2}\,dt + \cos x\left(3x^2e^{x^6} - 3e^{9x^2}\right).$$

6. Let R be the region bounded by the x-axis, $y = 2 - \frac{1}{2}x$, and $y = x - 3$.

(a) Using one or more integrals, express the volume of the solid of revolution that we get by revolving R about the x-axis. **Do not evaluate your volume expression**.



After making a good sketch of the situation, we find that the region is bounded below by the x-axis. We also find that the region is bounded above by $y = x - 3$ for $3 \le x \le \dfrac{10}{3}$, and is bounded above by $y = 2 - \dfrac{1}{2}x$ for $\dfrac{10}{3} \le x \le 4$.

Hence, the volume is

$$\text{volume} = \int_3^{10/3} \pi (x-3)^2 \, dx + \int_{10/3}^4 \pi \left(2 - \frac{1}{2}x\right)^2 \, dx.$$

(b) Using one or more integrals, express the volume of the solid of revolution that we get by revolving R about the y-axis. **Do not evaluate your volume expression**.

We find the y-coordinate of the point of intersection of the two lines is $y = \dfrac{1}{3}$.

Solving $y = x - 3$ for x, we find $x = y + 3$.

Solving $y = 2 - \dfrac{1}{2}x$ for x, we find $x = 4 - 2y$.

Hence, the volume is

$$\text{volume} = \int_0^{1/3} \pi((4-2y)^2 - (y+3)^2) \, dy.$$