Math 125 D Autumn 2023
Mid-Term Exam Number One
October 19, 2023
Solutions

1. Evaluate the following indefinite integrals.

(a) $\int \left(\sqrt{x}-3\right)\left(\sqrt{x}+1\right)\, dx$

$$= \int (x - 2\sqrt{x} - 3)\, dx = \frac{1}{2}x^2 - \frac{4}{3}x^{3/2} - 3x + C.$$

(b) $\int \frac{x^3+5x^2+2}{x^2}\, dx$

$$= \int \left(x + 5 + 2x^{-2}\right)\, dx = \frac{1}{2}x^2 + 5x - 2x^{-1} + C.$$

(c) $\int x^3\sqrt{x^2+4}\, dx$

Let $u = x^2+4$ so $du = 2x\, dx$ and $x\, dx = \frac{1}{2}\, du$. Then $x^2 = u - 4$ and

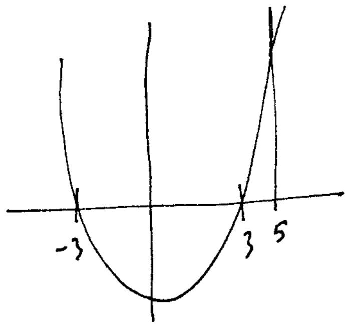
$$x^3\, dx = x^2 \cdot x\, dx = (u-4)\cdot \frac{1}{2}\, du = \frac{1}{2}(u-4)\, du.$$

Hence, the integral equals

$$\begin{aligned}\frac{1}{2}\int (u-4)\sqrt{u}\, du &= \frac{1}{2}\int (u^{3/2} - 4u^{1/2})\, du \\ &= \frac{1}{2}\left(\frac{2}{5}u^{5/2} - 4\cdot\frac{2}{3}u^{3/2}\right) + C \\ &= \frac{1}{5}(x^2+4)^{5/2} - \frac{4}{3}(x^2+4)^{3/2} + C.\end{aligned}$$

2. Evaluate the following definite integrals.

(a) $\int_0^5 |x^2 - 9|\,dx.$



$x^2 - 9 = 0$

$x^2 = 9$

$x = \pm 3$

By making a good sketch and finding roots, we find that $x^2 - 9 \geq 0$ when $x \geq 3$ and $x^2 - 9 < 0$ when $0 \leq x \leq 3$, so that

$$\begin{aligned}\int_0^5 |x^2 - 9|\,dx &= \int_0^3 (9 - x^2)\,dx + \int_3^5 (x^2 - 9)\,dx \\ &= (9x - \frac{1}{3}x^3)\Big|_0^3 + (\frac{1}{3}x^3 - 9x)\Big|_3^5 \\ &= \frac{98}{3}.\end{aligned}$$

(b) $\int_{-2}^{2} f(t)\,dt$ where $f(t) = g'(t)$ and $g(t) = te^{3t}$.

Since $f(t) = g'(t)$, $\int f(t)dt = g(t) + C$.

Hence,

$$\int_{-2}^{2} f(t)\,dt = g(t)\Big|_{-2}^{2} = g(2) - g(-2) = 2e^6 + 2e^{-6}.$$

(c) $\int_0^4 \frac{e^x}{e^x + 3}\,dx$

Let $u = e^x + 3$. Then $du = e^x dx$.

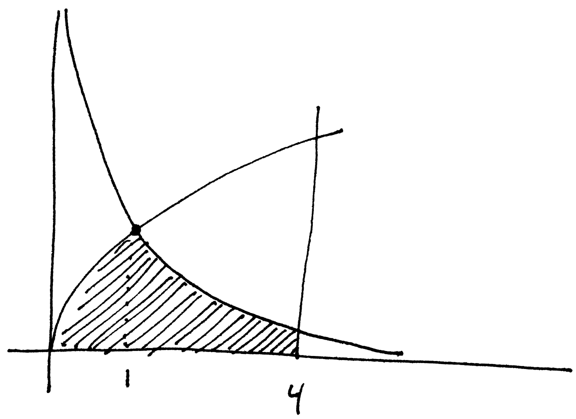
Hence,

$$\int \frac{e^x}{e^x + 3}\,dx = \int \frac{du}{u} = \ln|u| = \ln|e^x + 3|.$$

and

$$\int_0^4 \frac{e^x}{e^x + 3}\,dx = \ln|e^x + 3|\Big|_0^4 = \ln(e^4 + 3) - \ln(e^0 + 3) = \ln(e^4 + 3) - \ln 4.$$

3. Find the area of the region bounded by the curves $y = \dfrac{1}{x^3}$, $y = \sqrt{x}$, $x = 4$ and the x-axis.



$$\frac{1}{x^3} = \sqrt{x}$$
$$\frac{1}{x^6} = x$$
$$x^7 = 1$$
$$x = 1$$

After drawing a good sketch, we find that the two curves intersect at $x = 1$ with the region being bounded above by $y = \sqrt{x}$ and below by $y = 0$ for $0 \le x \le 1$, and the region being bounded above by $y = \dfrac{1}{x^3}$ and below by $y = 0$ for $1 \le x \le 4$.

Hence, the area of the region is

$$\begin{aligned}
\text{area} &= \int_0^1 \sqrt{x}\, dx + \int_1^4 \frac{1}{x^3}\, dx \\
&= \frac{2}{3} x^{3/2} \Big|_0^1 + \left(-\frac{1}{2}\right) x^{-2} \Big|_1^4 \\
&= \frac{2}{3} + \left(-\frac{1}{32} + \frac{1}{2}\right) \\
&= \frac{109}{96}.
\end{aligned}$$

4. You find yourself on a distant planet, where the acceleration due to gravity is not the same as on Earth.

   To measure the acceleration due to gravity, you perform an experiment.

   You construct a 60 meter tall tower. From the top of the tower, you throw a rock downward.

   The rock hits the ground exactly 8 seconds later.

   The final 10 meters of its fall takes exactly 1 second.

   What is the acceleration due to gravity on this distant planet?

   We begin with the assumption that acceleration is constant; let g be the acceleration due to gravity on this planet.

   Let $h(t)$ be height of the rock t seconds after it is thrown.

   Then
$$h''(t) = g$$
   so that
$$h'(t) = gt + C_1$$
   and
$$h(t) = \frac{1}{2}gt^2 + C_1 t + C_2$$
   for some constants C_1 and C_2, by integration.

   We know three things:

   - $h(0) = 60$
   - $h(8) = 0$
   - $h(7) = 10$ since the rock travels 10 meters in the 1 second before it reaches the ground.

   Hence,
$$h(0) = 60 = 0 + 0 + C_2$$
   so $C_2 = 60$, and
$$0 = 32g + 8C_1 + 60 \text{ and } 10 = 24.5g + 7C_1 + 60$$
   Using these last two equations and solving for g we find
$$g = -\frac{5}{7}$$
   so the acceleration due to gravity on this planet is $\frac{5}{7}$ m/s^2.

5. Let $g(x) = \sin x \int_{x^2}^{5x} e^{t^2}\, dt$.

Find $g'(x)$ (your answer may involve an integral or two).

By the Fundamental Theorem of Calculus, Part I, we know that if

$$f(x) = \int_a^x h(t)\, dt,$$

then $f'(x) = h(x)$. Hence, if

$$j(u) = \int_a^u h(t)\, dt,$$

and u is a function of x, then

$$\frac{d}{du} j(u)\, dt = h(u)$$

and, by the chain rule,

$$\frac{d}{dx} j(u) = \frac{d}{du} j(u) \cdot \frac{du}{dx}.$$

Thus,

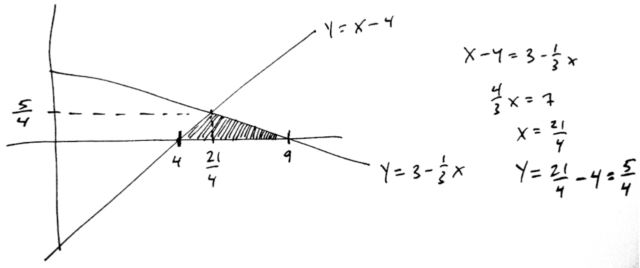
$$\begin{aligned} \frac{d}{dx} \int_{x^2}^{5x} e^{t^2}\, dx &= \frac{d}{dx} \int_0^{5x} e^{t^2}\, dx - \frac{d}{dx} \int_0^{x^2} e^{t^2}\, dx \\ &= e^{(5x)^2} \frac{d}{dx} 5x - e^{x^4} \frac{d}{dx} x^2 \\ &= 5e^{25x^2} - 2xe^{x^4}. \end{aligned}$$

Hence, using the product rule, we find

$$g'(x) = \cos x \int_{x^2}^{5x} e^{t^2}\, dt + (\sin x) \left(5e^{25x^2} - 2xe^{x^4} \right).$$

6. Let R be the region bounded by the x-axis, $y = 3 - \frac{1}{3}x$, and $y = x - 4$.

(a) Using one or more integrals, express the volume of the solid of revolution that we get by revolving R about the x-axis. **Do not evaluate your volume expression**.



After making a good sketch of the situation, we find that the region is bounded below by the x-axis. We also find that the region is bounded above by $y = x - 4$ for $4 \le x \le \dfrac{21}{4}$, and is bounded above by $y = 3 - \dfrac{1}{3}x$ for $\dfrac{21}{4} \le x \le 9$.

Hence, the volume is

$$\text{volume} = \int_4^{21/4} \pi(x-4)^2\,dx + \int_{21/4}^{9} \pi\left(3 - \frac{1}{3}x\right)^2\,dx.$$

(b) Using one or more integrals, express the volume of the solid of revolution that we get by revolving R about the y-axis. **Do not evaluate your volume expression**.

We find that the y-coordinate of the point of intersection of the two lines is $\frac{5}{4}$.
Also, if $y = 3 - \frac{1}{3}x$, then $x = 9 - 3y$, and if $y = x - 4$, then $x = y + 4$.
Hence, the volume is

$$\text{volume} = \int_0^{5/4} \left(\pi(9-3y)^2 - \pi(y+4)^2\right)dy.$$