Math 125G - Spring 2002
First Mid-Term Exam Solutions

1. Is $\frac{1}{2}x^2\ln x - \frac{1}{4}x^2$ an antiderivative of $x\ln x$? Explain.

   The derivative of $\frac{1}{2}x^2\ln x - \frac{1}{4}x^2$ is $x\ln x$ (details left to the reader, must be shown for credit), so, yes, $\frac{1}{2}x^2\ln x - \frac{1}{4}x^2$ an antiderivative of $x\ln x$.

2. Suppose $f''(x) = 2 + e^x$, $f'(0) = 3$ and $f(0) = 2$. Find $f(x)$.

   Since $f''(x) = 2 + e^x$, $f'(x) = \int (2 + e^x)\,dx = 2x + e^x + C$. Using the fact that $f'(0) = 3$, we have $f'(0) = 0 + e^0 + C = 1 + C = 3$, so $C = 2$, i.e., $f'(x) = 2x + e^x + 2$. From this, we get $f(x) = \int (2x + e^x + 2)\,dx = x^2 + e^x + 2x + D$, where D is an arbitrary constant. Using the fact that $f(0) = 2$, we have $2 = f(0) = 0 + 1 + 0 + D$, so $D = 1$. Hence, $f(x) = 1 + 2x + x^2 + e^x$.

3. Use the midpoint rule with $n = 3$ to approximate the integral

   $$\int_0^6 \ln(\sin x + 3)\,dx.$$

   Cutting the interval $[0, 6]$ into three equal subintervals, we find these subintervals have length 2 and have centers 1, 3 and 5. Hence the midpoint rule approximation is

   $$2\left(\ln(\sin 1 + 3) + \ln(\sin 3 + 3) + \ln(\sin 5 + 3)\right) = 6.40782354841865609....$$

   Note that your answer should have at least 4 correct digits, so be sure to use as many digits throughout your calculations as you can (8 is sufficient).

4. Solve the following equation for m:

   $$\int_0^1 f(x)\,dx - 2\int_0^{\frac{1}{2}} f(2x)\,dx - \int_1^0 f(x)\,dx = m\int_0^1 f(x)\,dx$$

   The third integral is equal to $-\int_0^1 f(x)\,dx$, so the left hand expression simplifies to

   $$2\int_0^1 f(x)\,dx - 2\int_0^{\frac{1}{2}} f(2x)\,dx.$$

Letting $u = 2x$, so that $du = 2dx$, we have

$$\int_0^{\frac{1}{2}} f(2x)\,dx = \frac{1}{2}\int_0^1 f(u)\,du = \frac{1}{2}\int_0^1 f(x)\,dx$$

Hence,

$$2\int_0^1 f(x)\,dx - 2\int_0^{\frac{1}{2}} f(2x)\,dx = 2\int_0^1 f(x)\,dx - \int_0^1 f(x)\,dx = \int_0^1 f(x)\,dx$$

and so $m = 1$.

5. Find the derivative of each of the following functions.

   (a) $g(x) = \int_2^{x^2} \sin(t^2 + 3t)\,dt$

   $g'(x) = \sin(x^4 + 3x^2)2x$

   (b) $h(x) = \int_2^3 \frac{\ln v}{\sin v}\,dv$

   $h'(x) = 0$ (note that $h(x)$ is a constant).

6. Evaluate the following integrals:

   (a) $\int \frac{x}{x^2+1}\,dx$

   Letting $u = x^2 + 1$, so $du = 2xdx$, we have

$$\int \frac{x}{x^2+1}\,dx = \int \frac{\frac{1}{2}du}{u} = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|x^2+1| + C = \frac{1}{2}\ln(x^2+1) + C.$$

   (b) $\int_{-1}^1 (2-x)^6\,dx$

   Letting $u = 2 - x$, so $du = -dx$, we have

$$\int_{-1}^1 (2-x)^6\,dx = -\int_3^1 u^6\,du = \int_1^3 u^6\,du = \left.\frac{u^7}{7}\right|_1^3 = \frac{3^7-1}{7} = \frac{2186}{7}.$$

7. Find the area of the region bounded by the curves $y = x^2 - \frac{3}{2}$ and $y = \frac{1}{2} - x^2$.

   We first have to find the intersection of the two curves. Setting them equal to each other we have $x^2 - \frac{3}{2} = \frac{1}{2} - x^2$, i.e. $2x^2 = 2$, i.e. $x^2 = 1$, so $x = -1$ and $x = 1$. Noting that at $x = 0$ $\frac{1}{2} - x^2 > x^2 - \frac{3}{2}$, we find the areas between the curves by evaluating the integral

$$\int_{-1}^{1} (\frac{1}{2} - x^2 - (x^2 - \frac{3}{2}))\, dx = \int_{-1}^{1} (2 - 2x^2)\, dx$$

$$= (2x - \frac{2}{3}x^3)\Big|_{-1}^{1} = (2 - \frac{2}{3} - (-2 + \frac{2}{3})) = 4 - \frac{4}{3} = \frac{8}{3}.$$

8. Let $p > 1$. Suppose the region in the first quadrant bounded by $y = x$ and $y = x^p$ is rotated about the x-axis to create a solid of revolution. If the volume of the solid is $\frac{\pi}{6}$, find p.

   If $x = x^p$ then $x = 0$ or $x = 1$, and $x^p \leq x$ for $0 \leq x \leq 1$ so the volume of this solid of revolution is given by

$$\int_{0}^{1} (\pi x^2 - \pi (x^p)^2)\, dx = \frac{\pi}{3}x^3 - \frac{\pi}{2p+1}x^{2p+1}\Big|_{0}^{1} = \frac{\pi}{3} - \frac{\pi}{2p+1}.$$

   For this to be equal to $\frac{\pi}{6}$, we must have

$$\frac{\pi}{3} - \frac{\pi}{2p+1} = \frac{\pi}{6}$$

   i.e.

$$\frac{\pi}{6} = \frac{\pi}{2p+1}$$

   so $2p + 1 = 6$, and hence $p = \frac{5}{2}$.