Math 125A version 1 - Spring 2003
Mid-Term Exam Number One Solutions
April 24, 2003

1. Find the derivative of each of the following functions.

   (a) $f(x) = \int_{-2}^{x^2} e^{t^3}\, dt.$
   Solution:
   $$f'(x) = e^{(x^2)^3} 2x = 2xe^{x^6}.$$

   (b) $g(x) = \int_{\ln x}^{\sin x} \sin(t^2)\, dt.$
   Solution:
   $$g'(x) = \sin((\sin x)^2)\cos x - \sin((\ln x)^2)\frac{1}{x}.$$

2. Evaluate the following integrals.

   (a) $\int x\sqrt{x+2}\, dx$
   Solution: Let $u = x + 2$ so $du = dx$ and $x = u - 2$. So
   $$\int x\sqrt{x+2}\, dx = \int (u-2)\sqrt{u}\, du = \int \left(u^{\frac{3}{2}} - 2u^{\frac{1}{2}}\right) du$$
   $$= \frac{2}{5}u^{\frac{5}{2}} - \frac{4}{3}u^{\frac{3}{2}} + C = \frac{2}{5}(x+2)^{\frac{5}{2}} - \frac{4}{3}(x+2)^{\frac{3}{2}} + C.$$

   (b) $\int \frac{dx}{x(\ln x)^3}$
   Solution: Let $u = \ln x$, so $du = \frac{1}{x}\, dx$. Then
   $$\int \frac{dx}{x(\ln x)^3} = \int \frac{du}{u^3} = \int u^{-3}\, du = \frac{u^{-2}}{-2} + C = -\frac{1}{2(\ln x)^2} + C.$$

   (c) $\int \frac{\sin x}{1+\cos^2 x}\, dx$
   Solution: Let $u = \cos x$, so $du = -\sin x\, dx$, i.e., $-du = \sin x\, dx$. Then
   $$\int \frac{\sin x}{1+\cos^2 x}\, dx = \int \frac{-du}{1+u^2} = -\tan^{-1} u + C = -\tan^{-1}(\cos x) + C.$$

3. Consider the region R in the first quadrant bounded by $y = x^2$, $y = -\frac{1}{2}x + 5$ and the x-axis. Find the volume of the solid of revolution created by revolving R about the x-axis.

   Solution: The intersection of $y = x^2$ with $y = -\frac{1}{2}x + 5$ in the first quadrant is $(2, 4)$, and the line $y = -\frac{1}{2}x + 5$ has x-intercept $x = 10$. So the volume of the solid asked for is
   $$\int_0^2 \pi (x^2)^2\, dx + \int_2^{10} \pi(-\frac{1}{2}x + 5)^2\, dx = \pi\frac{1}{5}x^5\Big|_0^2 + \pi\left(-\frac{2}{3}(-\frac{1}{2}x+5)^3\right)\Big|_2^{10}$$

$$= \pi\frac{2^5}{5} + \left(0 + \pi\frac{2}{3}4^3\right) = \frac{736}{15}\pi.$$

4. Consider the region S bounded by $y = \sqrt{x}$, $y = \frac{1}{2}x$, $x = 1$ and $x = 2$. Find the volume of the solid of revolution created by revolving S about the line $x = 4$.

   Solution: The volume is

$$2\pi\int_1^2 (4-x)(\sqrt{x} - \frac{1}{2}x)\, dx = 2\pi\int_1^2 \left(4x^{\frac{1}{2}} - 2x - x^{\frac{3}{2}} + \frac{1}{2}x^2\right) dx$$

$$= 2\pi\left(\frac{8}{3}x^{\frac{3}{2}} - x^2 - \frac{2}{5}x^{\frac{5}{2}} + \frac{1}{6}x^3\Big|_1^2\right) = 2\pi\left(\frac{8}{3}2^{\frac{3}{2}} - 4 - \frac{2}{5}2^{\frac{5}{2}} + \frac{1}{6}8 - \left(\frac{8}{3} - 1 - \frac{2}{5} + \frac{1}{6}\right)\right)$$

$$= 2\pi\left(\frac{8}{3}2^{\frac{3}{2}} - \frac{2}{5}2^{\frac{5}{2}} - \frac{41}{10}\right).$$

5. Find the value of $k > 0$ so that the region bounded by $y = x^k$ and $y = x^{1/k}$ has area $\frac{3}{4}$.

   Solution: Since $y = x^k$ and $y = x^{1/k}$ intersect at $x = 0$ and $x = 1$, the area bounded by them is, assuming $k > 1$,

$$\int_0^1 \left(x^{1/k} - x^k\right) dx = \frac{x^{1+\frac{1}{k}}}{1+\frac{1}{k}} - \frac{x^{k+1}}{k+1}\Bigg|_0^1 = \frac{1}{1+\frac{1}{k}} - \frac{1}{k+1} = \frac{k}{k+1} - \frac{1}{k+1} = \frac{k-1}{k+1}.$$

   Setting this equal to $\frac{3}{4}$ and solving we have:

$$\frac{k-1}{k+1} = \frac{3}{4}$$

$$k - 1 = \frac{3}{4}k + \frac{3}{4}$$

$$\frac{1}{4}k = \frac{7}{4}$$

   so $k = 7$. Because of the symmetry of the situation, $k = \frac{1}{7}$ is a solution also.

6. Use the Midpoint Rule with $n = 4$ to estimate the area of the region bounded by $y = \sin\left(\frac{1}{x}\right)$, $y = 0$, $x = 1$ and $x = 2$.

   Solution: Cutting the interval $[1, 2]$ into 4 equal subintervals gives us the subintervals $[1, 1.25]$,$[1.25, 1.5]$, $[1.5, 1.75]$, and $[1.75, 2]$. The midpoints of these intervals are 1.125, 1.375, 1.625, and 1.875. So the Midpoint Rule approximation is

$$\frac{1}{4}\left(\sin\left(\frac{1}{1.125}\right) + \sin\left(\frac{1}{1.375}\right) + \sin\left(\frac{1}{1.625}\right) + \sin\left(\frac{1}{1.825}\right)\right)$$

$$= 0.74314129743833220140....$$

7. An accident occurred at the Tasty Foods company, and a lot of radioactive gas was released. A tree nearby was severely affected, and it started growing at an unnatural rate. Research has shown that trees affected by this kind of radiation grow at a rate (in meters/day) given by

$$r(t) = at^2$$

where t is the time (in days) since the exposure to the radiation, and a is a positive constant.

Ten days after the radiation leak the tree was 5 meters tall. After 20 days it was 10 meters tall. When will the tree be 20 meters tall?

Solution: Using the fact that the integral of rate of change is total change, the height of the tree is given by

$$h(t) = h(0) + \int_0^t r(x)\,dx = h(0) + \int_0^t ax^2\,dx = h(0) + \frac{1}{3}at^3$$

where $h(0)$ is the height of the tree when it was irradiated. We know $h(10) = 5$ and $h(20) = 10$ so

$$5 = h(0) + \frac{a}{3}10^3$$

and

$$10 = h(0) + \frac{a}{3}20^3.$$

Subtracting, we get

$$5 = \frac{1}{3}a(20^3 - 10^3) = \frac{7000}{3}a$$

so $a = \frac{15}{7000} = \frac{3}{1400}$ and

$$h(0) = 5 - \frac{1000}{1400} = \frac{30}{7}.$$

Thus

$$h(t) = \frac{30}{7} + \frac{1}{1400}t^3.$$

Setting this equal to 20 and solving for t, we get

$$t = \left(1400(20 - \frac{30}{7})\right)^{\frac{1}{3}} = 22000^{\frac{1}{3}},$$

or about 28.02 days.