Math 125 D and H - Spring 2004
Mid-Term Exam Number One
Solutions
April 22, 2004

1. Evaluate each of the following indefinite integrals.

   (a) $\int \frac{x^2}{x^3+5}\,dx = \frac{1}{3}\ln|x^3+5| + C$

   (b) $\int (x^2+3)^2\,dx = \int (x^4+6x^2+9)\,dx = \frac{1}{5}x^5 + 2x^3 + 9x + C$

   (c) $\int x^3\sqrt{x^2+4}\,dx = \frac{1}{5}(x^2+4)^{5/2} - \frac{4}{3}(x^2+4)^{3/2} + C$

   (d) $\int \frac{1}{x\ln x}\,dx = \ln|\ln x| + C$

2. Alice falls from a plane at an altitude of 3000 meters. She falls in such a way that she is accelerating at a rate of

   $-9.8 + 0.3t \text{ m/s}^2$

   t seconds after the start of her fall. Assume her initial velocity is zero.

   (a) What is her velocity after 6 seconds?
   Her velocity will be the integral of her rate of acceleration from $t = 0$ to $t = 6$:

   $$v = \int_0^6 (-9.8 + 0.3t)\,dt = (-9.8t + 0.15t^2)|_0^6 = -9.8(6) + 0.15(6^2) = -53.4 \text{ m/s}.$$

   (b) How far off the ground will she be after falling for 6 seconds?
   Her velocity after t seconds is given by

   $v = -9.8t + 0.15t^2$

   Integrating this, we find her position is

   $h = -4.9t^2 + 0.05t^3 + C$

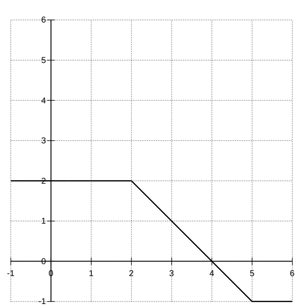
   Since $h = 3000$ when $t = 0$, $C = 3000$, so her height after t seconds is

   $h = -4.9t^2 + 0.05t^3 + 3000$

   and so after 6 seconds, she'll be

   $h = -4.9(6^2) + 0.05(6^3) + 3000 = 2834.4$ meters off the ground.

3. The graph of $f(x)$ is given below. Let $A(x) = \int_0^x f(t)\,dt$.



   Evaluate each of the following:

(a) $A(2) = (2)(2) = 4$

(b) $A'(3) = 1$

(c) $A(6) = (2)(2) + 2 - \frac{1}{2} - 1 = \frac{9}{2}$

(d) $A(4) - A(3) = \frac{1}{2}$

4. Let R be the region in the first quadrant bounded by $y = 2 - x^2$, $y = x^2$, and the y-axis.

   (a) Find the volume of the solid of revolution created by revolving R about the y-axis.
   The curves $y = x^2$ and $y = 2 - x^2$ intersect in the first quadrant at the point $(1, 1)$.

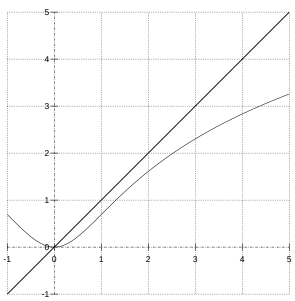
$$V = \int_0^1 2\pi x(2 - x^2 - x^2)\,dx = \int_0^1 2\pi(2x - 2x^3)\,dx = 2\pi\ (x^2 - \frac{1}{2}x^4)\Big|_0^1 = 2\pi(1 - \frac{1}{2}) = \pi.$$

   (b) Find the volume of the solid of revolution created by revolving R about the x-axis.

$$V = \int_0^1 \left(\pi(2 - x^2)^2 - \pi(x^2)^2\right)\,dx = \pi \int_0^1 \left(4 - 4x^2 + x^4 - x^4\right)\,dx$$

$$= \pi \int_0^1 (4 - 4x^2)\,dx = \pi\left(4x - \frac{4}{3}x^3\right)\Big|_0^1 = \pi\left(4 - \frac{4}{3}\right) = \frac{8}{3}\pi$$

5. Let R be the region bounded by $y = x$, $y = \ln(x^2 + 1)$, and $x = 3$. The curves are shown in the figure.

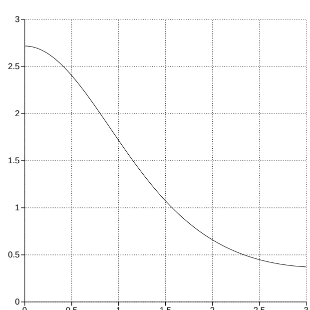


Set up an integral that gives the volume of the solid of revolution created by revolving R about the line $x = 5$. DO NOT EVALUATE THE INTEGRAL.

The method of cylindrical shells is the easiest way to go on this problem:

$$V = \int_0^3 2\pi(5 - x)(x - \ln(x^2 + 1))\,dx$$

6. Here is a graph of $y = e^{\cos x}$ on the interval $0 \le x \le 3$:



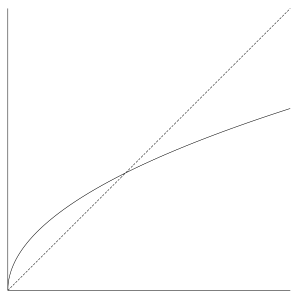
Use the midpoint rule with n=3 to approximate the value of the following integral:

$$\int_0^3 e^{\cos x}\,dx$$

$$\int_0^3 e^{\cos x}\,dx \approx (1)e^{\cos(0.5)} + (1)e^{\cos(1.5)} + (1)e^{\cos(2.5)} = 3.927193071...$$

Note that you must have your calculator in radian mode in order to correctly calculate $\cos(0.5) = 0.87758256...$ etc.

7. Find the value of m so that the region bounded by $y = \sqrt{x}$ and $y = mx$ has an area of 4.



The curves $y = \sqrt{x}$ and $y = mx$ intersect at $x = \dfrac{1}{m^2}$. The area of the region bounded by these curves is

$$\int_0^{1/m^2} \left(\sqrt{x} - mx\right)\,dx = \left(\frac{2}{3}x^{3/2} - \frac{1}{2}mx^2\right)\Big|_0^{1/m^2} = \frac{1}{6m^3}.$$

So, if the area equals 4, then we have

$$4 = \frac{1}{6m^3}$$

from which we find

$$m = \frac{1}{\sqrt[3]{24}} = \frac{1}{2\sqrt[3]{3}} = 0.3466806372...$$