

Math 125 C - Winter 2005
Mid-Term Exam Number One Solutions
January 27, 2005

1. (a) Is $\frac{e^x}{x}$ an antiderivative of $\frac{e^x}{x} \left(\frac{1}{x^2} - 1 \right)$? Explain.

Since

$$\frac{d}{dx} \frac{e^x}{x} = \frac{e^x}{x} \left(1 - \frac{1}{x} \right) \neq \frac{e^x}{x} \left(\frac{1}{x^2} - 1 \right)$$

the answer is no.

- (b) Suppose $f(t) = 3g(t) - 6$ and

$$\int_{-3}^5 g(t) dt = 12.$$

Find

$$\int_{-3}^5 f(x) dx.$$

$$\int_{-3}^5 f(x) dx = \int_{-3}^5 f(t) dt = \int_{-3}^5 (3g(t) - 6) dt$$

$$= 3 \int_{-3}^5 g(t) dt - \int_{-3}^5 6 dt = 3 \cdot 12 - 6(5 - (-3)) = -12.$$

2. Evaluate the following integrals:

(a)
$$\int_0^1 \left(x^5 - 3x^4 + \frac{1}{2}x^3 - x \right) dx$$
$$= \left(\frac{1}{6}x^6 - \frac{3}{5}x^5 + \frac{1}{8}x^4 - \frac{1}{2}x^2 \right) \Big|_0^1 = -\frac{97}{120} = -0.8083333333\dots$$

(b)
$$\int_{-5}^5 |3 + 5t| dt$$

The easiest way to evaluate this integral is to interpret it as an area. It is the sum of the areas of two triangles: one has width $-\frac{3}{5} - (-5)$ and height 22, the other has width $5 - (-\frac{3}{5})$ and height 28. Hence the integral is equal to

$$\frac{1}{2}(22) \left(-\frac{3}{5} + 5 \right) + \frac{1}{2}(28) \left(5 + \frac{3}{5} \right) = \frac{634}{5} = 126.8.$$

(c)
$$\int_e^{e^2} \frac{1}{x(\ln x)^4} dx$$

Let $u = \ln x$ so the $du = \frac{1}{x} dx$ and we have

$$\int_e^{e^2} \frac{1}{x(\ln x)^4} dx = \int_1^2 \frac{1}{u^4} du = -\frac{1}{3}u^{-3} \Big|_1^2 = -\frac{7}{24}.$$

3. Evaluate the following integrals:

(a) $\int \frac{1 + \cos x}{\sin x + x} dx$

Let $u = \sin x + x$ so that $du = (\cos x + 1) dx$ and the integral equals

$$\int \frac{du}{u} = \ln |u| + C = \ln |\sin x + x| + C.$$

(b) $\int e^x \sin(e^x - 3) dx$

Let $u = e^x - 3$ so that $du = e^x dx$. Then the integral equals

$$\int \sin u du = -\cos u + C = -\cos(e^x - 3) + C.$$

(c) $\int (x^3 + x) \sqrt{x^2 + 1} dx$

There are a number of ways to handle this integral. One is to expand the integrand to get

$$\int (x^3 + x) \sqrt{x^2 + 1} dx = \int x^3 \sqrt{x^2 + 1} dx + \int x \sqrt{x^2 + 1} dx = A + B, \text{ say.}$$

Using the same substitution for both integrals, let $u = x^2 + 1$ so $x^2 = u - 1$ and $du = 2x dx$, or $\frac{1}{2} du = x dx$. Then

$$\begin{aligned} A &= \frac{1}{2} \int (u - 1) \sqrt{u} du = \frac{1}{2} \int (u^{3/2} - u^{1/2}) du = \frac{1}{5} u^{5/2} - \frac{1}{3} u^{3/2} + C \\ &= \frac{1}{5} (x^2 + 1)^{5/2} - \frac{1}{3} (x^2 + 1)^{3/2} + C. \end{aligned}$$

and

$$B = \frac{1}{2} \int u^{1/2} du = \frac{1}{3} u^{3/2} + C = \frac{1}{3} (x^2 + 1)^{3/2} + C.$$

So, the integral equals

$$A + B = \frac{1}{5} (x^2 + 1)^{5/2} + C.$$

4. Consider the region bounded by $y = 10 - x^2$ and $y = x^2 - 2x + 6$.

(a) Find the area of the region.

Determining the intersection of the two curves we find

$$x^2 - 2x + 6 = 10 - x^2$$

$$x^2 - x - 2 = (x - 2)(x + 1) = 0$$

so the intersections are at $x = -1$ and $x = 2$. Evaluating the two functions at $x = 0$, we find that $10 - x^2$ is the "upper function", so the area is

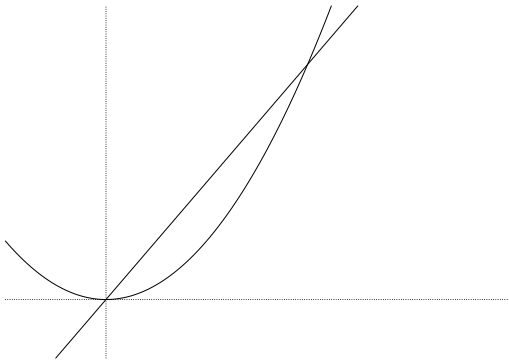
$$\int_{-1}^2 (10 - x^2 - (x^2 - 2x + 6)) dx = \int_{-1}^2 (4 + 2x - 2x^2) dx = \left(4x + x^2 - \frac{2}{3} x^3 \right) \Big|_{-1}^2 = 9.$$

- (b) Set up but DO NOT EVALUATE an integral representing the volume of the solid of revolution formed by revolving this region about the line $x = 5$.

Using the cylindrical shells method, the volume is

$$\int_{-1}^2 2\pi(5-x)(10-x^2-(x^2-2x+6)) dx = \int_{-1}^2 2\pi(5-x)(4+2x-2x^2) dx.$$

5. A region is bounded by two curves: $y = x^2$, and a line with a positive slope passing through the origin.



- (a) If the area of this region is 1, what is the slope of the line?

Let m be the slope of the line. Then the line is $y = mx$ and it intersects the curve when

$$mx = x^2$$

$$x = m \text{ or } x = 0.$$

Hence the area of the region is given by the integral

$$\int_0^m (mx - x^2) dx = \left(\frac{1}{2}mx^2 - \frac{1}{3}x^3 \right) \Big|_0^m = \frac{1}{6}m^3.$$

So,

$$\frac{1}{6}m^3 = 1$$

and so

$$m = 6^{1/3} = 1.81712\dots$$

- (b) If the volume of the solid of revolution created by revolving this region about the x -axis is 1, what is the slope of the line?

As in (a), the line is $y = mx$ and it intersects $y = x^2$ at $x = 0$ and $x = m$. So, using the "washer" method, the volume is given by

$$\int_0^m \pi((mx)^2 - (x^2)^2) dx = \pi \int_0^m \left(\frac{1}{3}m^2x^3 - \frac{1}{5}x^5 \right) \Big|_0^m = \frac{2}{15}\pi m^5 = 1.$$

Solving this last equation for m , we have

$$m = \left(\frac{15}{2\pi} \right)^{\frac{1}{5}} = 1.190096\dots$$