Math 125 A, F, G - Spring 2003
Mid-Term Exam Number Two Solutions
Version 1
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1. Consider the region in the first quadrant bounded by $y = x^{\frac{3}{2}}$, $x = 0$ and $y = 8$. Suppose this region is revolved about the y-axis to create a three-dimensional solid. Suppose we have a tank with the shape of that solid, oriented so that the y-axis is perpendicular to the ground, the origin is at the bottom of the tank, and units are in meters (so the tank is 8 meters tall). If the tank is filled with a liquid with density 2300 kg/m^3, how much work is required to pump all of the liquid to the top of the tank?

   Solution:

   The integral we need to evaluate is

$$\int_0^8 (8-y)\pi \left(y^{\frac{2}{3}}\right)^2 (2300)(g)\, dy.$$

   This is

$$2300\pi g \int_0^8 (8-y)y^{\frac{4}{3}}\, dy = 2300\pi g \int_0^8 \left(8y^{\frac{4}{3}} - y^{\frac{7}{3}}\right) dy = 2300\pi g \left(\frac{24}{7}y^{\frac{7}{3}} - \frac{3}{10}y^{\frac{10}{3}}\right)\Big|_0^8$$

$$= 2300\pi g \left(\frac{24}{7}2^7 - \frac{3}{10}2^{10}\right) = 2300\pi g \left(\frac{4608}{35}\right) \approx 9322839.562 \text{ Joules.}$$

2. For what $k > 0$ do $y = x^2$ and $y = 10 - x^2$ have the same average value on the interval $[0, k]$?

   Solution: The average value of x^2 on the interval $[0, k]$ is

$$\frac{1}{k}\int_0^k x^2\, dx = \frac{1}{k}\left(\frac{1}{3}x^3\right)\Big|_0^k = \frac{1}{k}\frac{1}{3}k^3 = \frac{1}{3}k^2.$$

   The average value of $10 - x^2$ on the interval $[0, k]$ is

$$\frac{1}{k}\int_0^k \left(10 - x^2\right) dx = \frac{1}{k}\left(10x - \frac{1}{3}x^3\right)\Big|_0^k = \frac{1}{k}\left(10k - \frac{1}{3}k^3\right) = 10 - \frac{1}{3}k^2.$$

   If the average values are equal, then

$$\frac{1}{3}k^2 = 10 - \frac{1}{3}k^2$$

   and if we solve that equation we find a single positive solution, $k = \sqrt{15}$.

3. Use Simpson's Rule with $n = 6$ to approximate the integral:

$$\int_2^5 \frac{1}{\ln x}\,dx$$

Maintain at least 4 digits of precision at all times.

Solution: Simpson's Rule applied to this integral gives us the approximation

$$\frac{\frac{5-2}{6}}{3}\left(\frac{1}{\ln 2} + \frac{4}{\ln 2.5} + \frac{2}{\ln 3} + \frac{4}{\ln 3.5} + \frac{2}{\ln 4} + \frac{4}{\ln 4.5} + \frac{1}{\ln 5}\right) = 2.5908350256170....$$

4. Evaluate each of the following integrals:

(a) $\displaystyle\int x^5 \ln x\,dx$

Solution: Using integration by parts, with $u = \ln x$ and $dv = x^5\,dx$, we have

$$\int x^5 \ln x\,dx = \frac{1}{6}x^6 \ln x - \int \frac{1}{6}x^5\,dx = \frac{1}{6}x^6 \ln x - \frac{1}{36}x^6 + C.$$

(b) $\displaystyle\int \sin^3 x \cos^6 x\,dx$

Solution:

$$I = \int \sin^3 x \cos^6 x\,dx = \int \sin x(1 - \cos^2 x)\cos^6 x\,dx$$

Let $u = \cos x$ so

$$I = -\int (1 - u^2)u^6\,du = \int (u^8 - u^6)\,du = \frac{1}{9}u^9 - \frac{1}{7}u^7 + C = \frac{1}{9}\cos^9 x - \frac{1}{7}\cos^7 x + C.$$

5. Evaluate each of the following integrals:

(a) $\displaystyle\int \frac{3\,dx}{x^2 + 3x - 10}$

Solution: Since

$$\frac{1}{x^2 + 3x - 10} = \frac{1}{(x+5)(x-2)} = \frac{-\frac{1}{7}}{x+5} + \frac{\frac{1}{7}}{x-2}$$

we have

$$\int \frac{3\,dx}{x^2 + 3x - 10} = -\frac{3}{7}\ln|x+5| + \frac{3}{7}\ln|x-2| + C.$$

(b) $\displaystyle\int \frac{5\,dx}{x^2 + 8x + 20}$

Solution: Since $x^2 + 8x + 20 = (x+4)^2 + 4$, we can use the substitution $x + 4 = 2\tan\theta$:

$$\int \frac{5\,dx}{x^2 + 8x + 20} = \int \frac{10\sec^2\theta\,d\theta}{4\tan^2\theta + 4} = \int \frac{5}{2}\,d\theta = \frac{5}{2}\theta + C = \frac{5}{2}\tan^{-1}\left(\frac{x+4}{2}\right) + C.$$

6. Evaluate the following integrals.

(a) $\displaystyle\int \frac{dx}{\sqrt{x^2 - 8x + 18}}$

Solution: Since $x^2 - 8x + 18 = (x-4)^2 + 2$, we can use the substitution $x - 4 = \sqrt{2}\tan\theta$:

$$\int \frac{dx}{\sqrt{x^2 - 8x + 18}} = \int \frac{dx}{\sqrt{(x-4)^2 + 2}} = \int \frac{\sqrt{2}\sec^2\theta\, d\theta}{\sqrt{2\tan^2\theta + 2}}$$

$$= \int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + C = \ln\left|\sqrt{x^2 - 8x + 18} + \frac{x-4}{\sqrt{2}}\right| + C.$$

(b) $\displaystyle\int \frac{dx}{x^3 + 3x^2}$

Solution: Since

$$\frac{1}{x^3 + 3x^2} = \frac{1}{x^2(x+3)} = \frac{-\frac{1}{9}}{x} + \frac{\frac{1}{3}}{x^2} + \frac{\frac{1}{9}}{x+3}$$

we have

$$\int \frac{dx}{x^3 + 3x^2} = -\frac{1}{9}\ln|x| - \frac{1}{3x} + \frac{1}{9}\ln|x+3| + C.$$

7. Evaluate the following integrals.

(a) $\displaystyle\int x^3 e^{x^2}\, dx$

Solution: Let $u = x^2$, so that $du = 2x\, dx$ and we have

$$\int x^3 e^{x^2}\, dx = \int x^2 e^{x^2}\, x\, dx = \int u e^u \frac{1}{2}\, du = \frac{1}{2}\int u e^u\, du.$$

Applying integration by parts, with $y = u, dz = e^u\, du$, we have

$$\frac{1}{2}\int u e^u\, du = \frac{1}{2}\left(u e^u - \int e^u\, du\right) = \frac{1}{2}\left(x^2 e^{x^2} - e^{x^2}\right) + C.$$

(b) $\displaystyle\int \frac{dx}{(x^2-1)^{\frac{5}{2}}}$

Solution: Using the substitution $x = \sec\theta$, we have

$$I = \int \frac{dx}{(x^2-1)^{\frac{5}{2}}} = \int \frac{\sec\theta}{\tan^4\theta}\, d\theta = \int \frac{\cos^3\theta}{\sin^4\theta}\, d\theta = \int \frac{\cos\theta(1 - \sin^2\theta)}{\sin^4\theta}\, d\theta$$

If we let $u = \sin\theta$, we get

$$I = \int \frac{1-u^2}{u^4}\, du = \int \left(\frac{1}{u^4} - \frac{1}{u^2}\right) du = -\frac{1}{3}u^{-3} + u^{-1} + C = -\frac{1}{3\sin^3\theta} + \frac{1}{\sin\theta} + C$$

$$= -\frac{x^3}{3(x^2-1)^{\frac{3}{2}}} + \frac{x}{(x^2-1)^{\frac{1}{2}}} + C.$$