Math 125 D and H - Spring 2004
Mid-Term Exam Number Two
Solutions
May 13, 2004

1. Evaluate each of the following integrals.

(a)
$$\int \frac{dx}{x^2 - 8x + 34} = \int \frac{dx}{(x-4)^2 + 18}$$

Let $u = x - 4$, so $du = dx$. Then the integral is

$$\int \frac{du}{u^2 + 18}$$

Let $u = \sqrt{18}\tan\theta$, $du = \sqrt{18}\sec^2\theta\, d\theta$. The integral is

$$\int \frac{\sqrt{18}\sec^2\theta}{18\sec^2\theta}\, d\theta = \int \frac{1}{\sqrt{18}}\, d\theta = \frac{1}{\sqrt{18}}\theta + C = \frac{1}{\sqrt{18}}\tan^{-1}\left(\frac{x-4}{\sqrt{18}}\right) + C.$$

(b)
$$\int e^{-3x}\sin 4x\, dx$$

Using integration by parts with

$$u = e^{-3x}, dv = \sin 4x\, dx$$

$$du = -3e^{-3x}\, dx, v = -\frac{1}{4}\cos 4x$$

we have

$$I = \int e^{-3x}\sin 4x\, dx = -\frac{1}{4}e^{-3x}\cos 4x - \frac{3}{4}\int e^{-3x}\cos 4x\, dx$$

Using integration by parts again with

$$u = e^{-3x}, dv = \cos 4x\, dx$$

$$du = -3e^{-3x}dx, v = \frac{1}{4}\sin 4x$$

we get

$$I = -\frac{1}{4}e^{-3x}\cos 4x - \frac{3}{4}\left(\frac{1}{4}e^{-3x}\sin 4x + \frac{3}{4}I\right)$$

so that

$$I = \frac{16}{25}\left(-\frac{1}{4}e^{-3x}\cos 4x - \frac{3}{16}e^{-3x}\sin 4x\right) + C$$

2. Evaluate each of the following integrals.

(a) $$\int \tan^4 x \sec^4 x \, dx = \int \tan^4 x \sec^2 x \sec^2 x \, dx = \int \tan^4 x (\tan^2 x + 1) \sec^2 x \, dx$$

$$= \int \left(\tan^6 x + \tan^4 x \right) \sec^2 x \, dx = \frac{1}{7} \tan^7 x + \frac{1}{5} \tan^5 x + C$$

(b) $$\int \frac{\cos x}{\sin^2 x + 1} \, dx$$

Let $u = \sin x$ so $du = \cos x \, dx$ to get

$$\int \frac{\cos x}{\sin^2 x + 1} \, dx = \int \frac{du}{u^2 + 1} = \tan^{-1}(\sin x) + C$$

3. Evaluate each of the following integrals.

(a) $$\int \frac{x^3}{\sqrt{x^2 - 4}} \, dx$$

Let $x = 2 \sec \theta$ so that $dx = 2 \sec \theta \tan \theta \, d\theta$ to get

$$\int \frac{x^3}{\sqrt{x^2 - 4}} \, dx = \int \frac{8 \sec^3 \theta \cdot 2 \sec \theta \tan \theta \, d\theta}{2 \tan \theta} = 8 \int \sec^4 \theta \, d\theta$$

$$= 8 \int \sec^2 \theta \sec^2 \theta \, d\theta = 8 \int (\tan^2 \theta + 1) \sec^2 \theta \, d\theta$$

$$= 8 \left(\frac{1}{3} \tan^3 \theta + \tan \theta \right) + C = \frac{8}{3} \left(\frac{\sqrt{x^2 - 4}}{2} \right)^3 + 8 \frac{\sqrt{x^2 - 4}}{2} + C.$$

(b) $$\int \frac{x^3 + 5x^2 + 1}{x^2 + 2x} \, dx$$

Using long division,

$$\frac{x^3 + 5x^2 + 1}{x^2 + 2x} = x + 3 + \frac{-6x + 1}{x(x+2)}$$

By partial fractions,

$$\frac{-6x + 1}{x(x+2)} = \frac{1/2}{x} + \frac{-13/2}{x + 2}$$

so that

$$\int \frac{x^3 + 5x^2 + 1}{x^2 + 2x} \, dx = \int x + 3 + \frac{1/2}{x} + \frac{-13/2}{x + 2} = \frac{1}{2} x^2 + 3x + \frac{1}{2} \ln |x| - \frac{13}{2} \ln |x + 2| + C$$

4. Evaluate each of the following integrals.

(a) $$\int_1^\infty \frac{\ln x}{x^6}\,dx$$

Using integration by parts,

$$\int \frac{\ln x}{x^6}\,dx = -\frac{1/5\ln x}{x^5} - \frac{1}{25}x^{-5} + C$$

so that

$$\int_1^\infty \frac{\ln x}{x^6}\,dx = \lim_{t\to\infty}\left(-\frac{1/5\ln t}{t^5} - \frac{1}{25t^5} - (0 - \frac{1}{25})\right) = (0 - 0 - (0 - \frac{1}{25})) = \frac{1}{25}$$

(b) $$\int_0^\infty \frac{x}{(x^2+5)^2}\,dx$$

Letting $u = x^2 + 5$ so that $du = 2x\,dx$, we have

$$\int \frac{x}{(x^2+5)^2}\,dx = \int \frac{1/2}{u^2}\,du = -\frac{1/2}{u} + C = -\frac{1/2}{x^2+5} + C$$

Hence,

$$\int_0^\infty \frac{x}{(x^2+5)^2}\,dx = \lim_{t\to\infty}\left(\frac{-1/2}{t^2+5} - \frac{-1/2}{5}\right) = \frac{1}{10}$$

5. Evaluate each of the following integrals.

(a) $$\int \frac{1}{2x^2-32}\,dx$$

By partial fractions,

$$\frac{1}{2x^2-32} = \frac{1/16}{x-4} - \frac{1/16}{x+4}$$

so that

$$\int \frac{1}{2x^2-32}\,dx = \int \left(\frac{1/16}{x-4} - \frac{1/16}{x+4}\right)\,dx = \frac{1}{16}\ln|x-4| - \frac{1}{16}\ln|x+4| + C$$

(b) $$\int \sin^3 x\cos^7 x\,dx = \int (1-\cos^2 x)\cos^7 x\sin x\,dx$$

$$= \int (\cos^7 x - \cos^9 x)\sin x\,dx = -\frac{1}{8}\cos^8 x + \frac{1}{10}\cos^{10} x + C$$

6. Suppose Matt dug a conical hole in the ground. The top of the hole is a circle 10 feet in diameter, and the hole is 8 feet deep.

   Suppose dirt has a density of 60 lb/ft^3. Set up but DO NOT EVALUATE an integral representing the amount of work Matt did to lift the dirt to the top of the hole.

   One possible answer:

$$\text{Work} = \int_{-8}^{0} 60(-y)\pi\left(\frac{y+8}{8/5}\right)^2\,dy$$