Hints and Partial Solutions to Practice Integrals

1. $\int \sin^4 x \cos^3 x\, dx$

$$= \int \sin^4 x \cos^2 x \cos x\, dx = \int \left(\sin^4 x\right)(1 - \sin^2 x) \cos x\, dx$$

Now let $u = \sin x$.

2. $\int \frac{dx}{\sqrt{6x - x^2}} = \int \frac{dx}{\sqrt{9 - (x-3)^2}}$. Let $x - 3 = 3\sin u$ so the integral is

$$\int \frac{3\cos u\, du}{\sqrt{9 - 9\sin^2 u}} = \int \frac{\cos u}{\cos u}\, du = u + C = \sin^{-1}\left(\frac{1}{3}x - 1\right) + C.$$

3. $\int \frac{\cos x\, dx}{\sqrt{1 + \sin x}}$ Let $u = 1 + \sin x$. Then the integral is $\int \frac{du}{\sqrt{u}} = \ldots$

4. $\int \frac{dx}{\sqrt{(a^2 + x^2)^3}}$

Let $x = a\tan u$. Then the integral is

$$\int \frac{a\sec^2 u}{(a^2\sec^2 u)^{\frac{3}{2}}}\, du = \int \frac{a\sec^2 u}{a^3\sec^3 u}\, du = \frac{1}{a^2}\int \frac{1}{\sec u}\, du = \frac{1}{a^2}\int \cos u\, du = \ldots$$

5. $\int \ln\sqrt{1 + x^2}\ dx = \frac{1}{2}\int \ln(1 + x^2)\, dx$. Use integration by parts with $u = \ln(1+x^2), dv = dv$, so we get

$$\frac{1}{2}\int \ln(1 + x^2)\, dx = \frac{1}{2}\left(x\ln(1 + x^2) - \int \frac{2x^2}{1 + x^2}\, dx\right)$$

Use the partial fractions method on this last integral, with

$$\frac{2x^2}{1 + x^2} = 2 - \frac{2}{x^2 + 1}.$$

6. $\int \frac{\sin^{-1} x\ dx}{\sqrt{1 - x^2}}$ Let $u = \sin^{-1} x$ so $du = \frac{dx}{\sqrt{1 - x^2}}$. Then the integral is

$$\int u\, du = \frac{1}{2}u^2 + C = \frac{1}{2}(\sin^{-1} x)^2 + C.$$

7. $\int \frac{(3x - 7)\ dx}{(x-1)(x-2)(x-3)}$ Use the partial fractions method to find constants A, B, and C so that

$$\frac{3x - 7}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$$

8. $\int e^{\ln\sqrt{x}}\ dx = \int \sqrt{x}\, dx = ...$

9. $\int x^2 e^x\ dx$ Use integration by parts twice.

10. $\int \frac{\tan x\ dx}{\cos^2 x} = \int \tan x \sec^2 x\, dx$. Let $u = \tan x$...

11. $\int \frac{\cos\sqrt{x}\ dx}{\sqrt{x}}$ Let $u = \sqrt{x}$.

12. $\int \frac{(x+1)\, dx}{x^2\,(x-1)}$ Use the partial fractions method to show that

$$\frac{x+1}{x^2(x-1)} = \frac{-2}{x} - \frac{1}{x^2} + \frac{2}{x-1}$$

and go from there.

13. $\int \tan^{-1} 10x\, dx$ Let $\theta = 10x$. Then the integral equals

$$\frac{1}{10}\int \tan^{-1}\theta\, d\theta.$$

Use integration by parts with $u = \tan^{-1}\theta$ and $dv = d\theta$ to get

$$\frac{1}{10}\left(\theta\tan^{-1}\theta - \int \frac{\theta\, d\theta}{\theta^2+1}\right)...$$

14. $\int \frac{dx}{(x^2-1)^{3/2}}$ Let $x = \sec\theta$. Then the integral equals

$$\int \frac{\sec\theta\tan\theta\, d\theta}{\tan^3\theta} = \int \frac{\sec\theta}{\tan^2\theta}\, d\theta = \int \frac{\cos\theta}{\sin^2\theta}\, d\theta.$$

Now let $w = \sin\theta$.

15. $\int x^{2/3}\left(x^{5/3}+1\right)^{2/3}\ dx$ Let $u = x^{\frac{5}{3}}$ so the integral becomes

$$\frac{3}{5}\int (u+1)^{\frac{2}{3}}\, du = ...$$

16. $\int \tan^3 3x \sec 3x\, dx = \int \tan^2 3x \sec 3x \tan 3x\, dx = \int (\sec^2 3x - 1)\sec 3x \tan 3x\, dx.$
Let $u = \sec 3x$. Then the integral becomes

$$\frac{1}{3}\int (u^2-1)\, du = ...$$

17. $\int e^x \cos 2x\, dx$ Use integration by parts twice.

18. $\int \frac{4x+1}{x^3+4x}\, dx$
Use partial fractions to find A, B and C so that

$$\frac{4x+1}{x^3+4x} = \frac{A}{x} + \frac{Bx+C}{x^2+4} = \frac{A}{x} + \frac{Bx}{x^2+4} + \frac{C}{x^2+4}$$

then integrate, using a trig. substitution on the last term.

19. $\int \frac{dx}{x^2+4x+8} = \int \frac{dx}{(x+2)^2+4}$ Use a trig. substitution.

20. $\int x^2 \sin x\, dx$ Use integration by parts twice.

21. $\int \sin^4 x \cos^2 x\, dx = \int \left(\frac{1}{2}(1-\cos 2x)\right)^2 \left(\frac{1}{2}(1+\cos 2x)\right) dx = ...$

22. $\int \frac{x^5\, dx}{\sqrt{1+x^2}}$ Use $x = \tan\theta$ to change the integral to

$$\int \tan^5\theta \sec\theta\, d\theta = \int \tan^4\theta \sec\theta \tan\theta\, d\theta = \int (\sec^2\theta - 1)^2 \sec\theta \tan\theta\, d\theta$$

Now let $u = \sec\theta$...

23. $\int \frac{\cos x\, dx}{\sin^3 x - \sin x}$ Let $u = \sin x$. The integral becomes

$$\int \frac{du}{u^3-u} = \int \frac{du}{u(u^2-1)} = \int \frac{du}{u(u+1)(u-1)}$$

Now use partial fractions.

Alternatively, write $\sin^3 x - \sin x = \sin x(\sin^2 x - 1) = -\sin x \cos^2 x$

and go from there.

Either way, the end result will be $-\ln|\tan x| + C$.

24. $\int x \tan^2 x\, dx = \int x(1-\sec^2 x)\, dx = \int x\, dx - \int x \sec^2 x\, dx$. Use integration by parts on the second integral.

25. $\int \tan^2 x \sec^2 x\, dx$ Let $u = \tan x$ so that the integral equals

$$\int u^2\, du = \frac{1}{3}u^3 + C = \frac{1}{3}\tan^3 x + C.$$

26. $\int x \ln\sqrt{x+2}\, dx = \frac{1}{2}\int x \ln(x+2)\, dx$. Use integration by parts.

27. $\displaystyle\int \frac{x\,dx}{x^4-16} = \int \frac{x}{(x^2-4)(x^2+4)}\,dx = \int \frac{x}{(x-2)(x+2)(x^2+4)}\,dx$. Use partial fractions.

28. $\displaystyle\int \frac{x^3+x^2}{x^2+x-2}\,dx = \int x + \frac{2x}{(x+2)(x-1)}\,dx$. Use partial fractions.

29. $\displaystyle\int \frac{dx}{x^{1/5}} = \int x^{-\frac{1}{5}}\,dx = \frac{5}{4}x^{\frac{4}{5}} + C.$

30. $\displaystyle\int \sin x \cos 2x\,dx$ Use the identity $\cos^2 x = \frac{1}{2}(1+\cos 2x)$ to write $\cos 2x = 2\cos^2 x - 1$ and the integral becomes

$$\int \sin x(2\cos^2 x - 1)\,dx = 2\int \cos^2 x \sin x\,dx - \int \sin x\,dx = ...$$

31. $\displaystyle\int \frac{dx}{x^4+4x^2+3} = \int \frac{dx}{(x^2+3)(x^2+1)}.$
Use the partial fractions method to write

$$\frac{1}{(x^2+3)(x^2+1)} = \frac{Ax+B}{x^2+3} + \frac{Cx+D}{x^2+1}$$

Find the constants and integrate.

32. $\displaystyle\int \frac{x^3\,dx}{\sqrt{16-x^2}}$ Let $x = 4\sin\theta$ so the integral becomes

$$\int \frac{64\sin^3\theta}{4\cos\theta} 4\cos\theta\,d\theta = 64\int \sin^3\theta\,d\theta = ...$$

33. $\displaystyle\int \sin^3 2x\cos^3 2x\,dx = \int \sin^3 2x\cos^2 2x\cos 2x\,dx = \int (\sin^3 2x)(1-\sin^2 2x)\cos 2x\,dx.$
Let $u = \sin 2x$...

34. $\displaystyle\int \frac{dx}{x\,(2+\ln x)}$ Let $u = \ln x$ so the integral becomes

$$\int \frac{du}{2+u} = \ln|2+u| + C = \ln|2+\ln x| + C.$$

35. $\displaystyle\int \frac{\cot x\,dx}{\ln(e\sin x)} = \int \frac{\cos x\,dx}{\sin x\ln(e\sin x)}.$ Let $u = \sin x$ so the integral becomes

$$\int \frac{du}{u\ln(eu)}.$$

Now let $v = \ln eu$ so the integral becomes

$$\int \frac{dv}{v} = \ln|v| + C = \ln|\ln(e\sin x)| + C$$

36. $\displaystyle\int \frac{\sec^2 x\,dx}{\sec^2 x - 3\tan x - 1} = \int \frac{\sec^2 x\,dx}{\tan^2 x - 3\tan x}$. Let $u = \tan x$ to get

$$\int \frac{du}{u^2 - 3u} = \int \frac{du}{u(u-3)}$$

and use partial fractions.