Math 125 Summary

- 4.10 - Antiderivatives
  - You should know what it means for $f(x)$ to be an antiderivative of $g(x)$.
  - Given two functions, $f(x)$ and $g(x)$, you should be able to say whether or not $f(x)$ is an antiderivative of $g(x)$.
  - How many antiderivative does a function have?
  - What is that "$+C$" business all about?
- 5.1 - Areas and Distances
  - How can we approximate the area of a region in the plane?
  - What is an interpretation of the area under the graph of a velocity function?
- 5.2 - The Definite Integral
  - You should understand the definition of the *definite integral* and its relation to *area* under a curve.
  - You should be able to use the *midpoint* rule to approximate a definite integral.
- 5.3 - The Fundamental Theorem of Calculus
  - Part 1: If f is continuous on $[a, b]$, then

    $$g(x) = \int_a^x f(t)\,dt$$

    is continuous on $[a, b]$ and differtiable on (a, b), $g'(x) = f(x)$.
  - Part 2: If f is continuous on $[a, b]$, then

    $$\int_a^b f(x)\,dx = F(b) - F(a)$$

    where F is any antiderivative of f.
  - Be sure you can differentiate functions like

    $$g(x) = \int_{\sin x}^{x^3} e^{t^2}\,dt$$

    using the chain rule and part 1 of the FTOC.
- 5.4 - Indefinite Integrals and the Net Change Theorem

- Here we get the notation that
$$\int f(x)\,dx$$
stands for the most general antiderivative of f.
- If $f'(x)$ is the rate of change of a quantity $f(x)$, then
$$\int_a^b f'(x)\,dx = f(b) - f(a)$$
yields the **net change** in $f(x)$ from $x = a$ to $x = b$.

- 5.5 - The Substitution Rule
  - The substitution rule is the most important and powerful tool for finding antiderivatives. It can be considered, to a certain extent, the reverse of the chain rule for differentiation.
  - Substitution is a way of transforming an indefinite integral into a new, equivalent indefinite integral. When trying to find antiderivatives, we may need to try several different substitutions until hitting on one that improves the integral we are working with to the point that we can find the antiderivative. Sometimes, more than one substitution, used in sequence, is an effective way to go. Practice will improve your ability to *see* the right substitutions.
  - As we get more techniques for finding antiderivatives, the substitution method will always be with us. It will pay to make sure you can use the method well now.

- 6.1 - Areas between Curves
  - The first of our many applications of the integral is to find the area between curves.
  - "Area is the integral of width."
  - In many instances you will want to express the area as an integral in y rather than in x.
  - You should always begin with a sketch of the region. Practice doing this by hand, since you will not have graphing devices, apps or the internet available during exams.
  - Note that
$$\int_a^b (f(x) - g(x))\ dx = -\int_a^b (g(x) - f(x))\ dx$$
so if your answer comes out negative (which is impossible for an area) check that you haven't got the difference of the two functions in the wrong order.

- 6.2 - Volumes (by Cross Section)

- Here is developed the idea that "volume is the integral of (cross sectional) area".
- Although many of the examples we looked at involve *solids of revolution* whose cross-sections are circles, this method applies to any solid that has cross sections whose area can be expressed as a function of x (or y).

- 6.3 - Volume by Cylindrical Shells
  - The method of washers/disks is great, but in certain cases we can result in an integral we are unable to evaluate, or, indeed, to setup. So we have another method: the method of cylindrical shells.
  - Even if the washer/disk method works, the cylindrical shells method can be easier. Practice will help you decide which method to use on a given solid.

- 6.4 - Work
  - Many problems finding the work required to perform a certain task can be solved by cutting the task into pieces, approximating the work to perform each piece, summing these approximations, and taking a limit as the number of pieces goes to infinity. The result is, of course, an integral.
  - Two popular categories of problems you might see on the final exam: cable problems and and pumping/digging problems. You should practice some of each type.

- 6.5 - Average Value of a Function
  - This is a very short section. You should understand the definition of the average value of a function on an interval.
  - The average value of $f(x)$ on the interval $a \leq x \leq b$ is

$$\frac{1}{b-a}\int_a^b f(x)\,dx$$

- 7.1 - Integration by Parts
  - You should understand how to apply the *integration by parts* technique:

$$\int u\,dv = uv - v\int du$$

  - You should be able to recognize integrands for which this technique is particularly appropriate, such as:
    1. a positive integer power of x times $\sin x$, $\cos x$, e^x (or related functions)
    2. $\ln x$ times any power of x

3. e^x times $\sin x$ or $\cos x$

- Integration by parts can be the method of last resort: if nothing else works, you can almost always try "parts": if you can differentiate the integrand, you can use it. Whether it helps or not is another question, but it works well with such function as $\ln x$ and $\arcsin x$ (or powers of x times one of these).

- 7.2 - Trigonometric Integrals

  - You should be able to integrate integrals of the form

$$\int \sin^n x \cos^m x \, dx$$

  and

$$\int \tan^n x \sec^m x \, dx$$

  - The strategies for these integrals depend on the *parities* of m and n (that is, whether they are even or odd). You should practice all cases. Note that the case where the power of $\tan x$ is even and the power of $\sec x$ is odd does not have a clear strategy, so a variety of techniques may be needed.

- 7.3 - Trigonometric Substitution

  - This technique exploits trigonometric Pythagorean identities to give useful substitutions that convert quadratic expressions into squares of trig functions. This makes it particularly useful for eliminating square roots (i.e., when you see a quadratic expression inside a square root, there's a good chance that a trig substitution might be useful).
  - One thing that makes this technique more work than a "simple" substitution is the work required to convert back to the original variable once an antiderivative is found. One efficient way to do this is the "triangle method" as described in lecture and the text. Be sure to practice this aspect of this technique.

- 7.4 - Partial Fractions

  - This technique is applicable to integrands that are rational functions.
  - You should know how to apply this technique to any rational function with a denominator that is factorable as a product of linear and distinct quadratic factors
  - The idea behind this method is completely algebraic: a rational function whose denominator is a product of linear factors can be expressed as the sum of simpler rational functions, each of which has a denominator which is linear or quadratic, and a constant or linear numerator. Such simpler functions are easily integrated.

- **Long division of polynomials** is a necessary first step when the numerator has degree equal to or greater than the degree of the denominator.

- 7.5 - Strategies for Integration
  - This section is perhaps the most important. On the exam, the integrals will simply be presented to you for you to solve. You will have to decide which technique, or techniques, to apply to find the antiderivative.

- 7.7 - Numerical Integration
  - When we want to evaluate a definite integral, and we cannot find an antiderivative of the integrand, we can approximate the value of the integral. Three techniques we've seen for doing this are
    * Midpoint Rule
    * Trapezoid Rule
    * Simpson's Rule

    All three methods are quite similar, and require us to "sample" the function at a number of equally spaced values of x, then combine the function values according to a certain formula which depends on the method.

    Note the two distinct uses of these methods: applied to an integral of an explicitly given function for which we cannot find an antiderivative, and to an integral of a function given only through a table of values. The implementation of the method is identical in both cases.

- 7.8 - Improper Integrals
  - There are two types of improper integrals:
    1. Those of the form
    $$\int_a^\infty f(x)\,dx, \int_{-\infty}^a f(x)\,dx, \text{or} \int_{-\infty}^\infty f(x)\,dx$$
    2. Those of the form
    $$\int_a^b f(x)\,dx$$
    where $f(x)$ is discontinuous somewhere on the interval $[a, b]$ (here, we allow the possibility that a and/or b equals ∞ or $-\infty$).
  - We treat improper integrals by defining them as the limits of "proper" integrals. For instance,
  $$\int_a^\infty f(x)\,dx = \lim_{k\to\infty} \int_a^k f(x)\,dx$$

- Be sure you remember l'Hospital's rule, since it is often the tool needed to evaluate the limits resulting from improper integrals.

- 8.1 - Arc Length

  - We can express the arc length S of a graph $y = f(x)$ over the interval $a \leq x \leq b$ via an integral:
  $$S = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$
  - For the vast majority of functions f we encounter, the above integral will not be one we can evaluate exactly, since we will be unable to find the antiderivative of the integrand. So, arc length problems on exams tend to be of two types: either they involve a cleverly chosen function for which the integrand is anti-differentiable, or you will be asked to approximate the arc length (using, e.g., Simpson's rule).

- Center of Mass

  - You should be able to determine the center of mass (also known as the centroid) of a given planar region.
  For a region in the plane bounded by $x = a, x = b, y = f(x)$ and $y = g(x)$ (with $f(x) \geq g(x)$), we can find the centroid by finding the moments of the region with respect to the two axes:
  $$M_y = \int_a^b x(f(x) - g(x))\, dx$$
  $$M_x = \int_a^b \frac{1}{2}\left(f(x)^2 - g(x)^2\right)\, dx$$
  These measure how the region is distributed with respect to the axes: if the region is mostly close to the y-axis, then M_y will be small, for example.
  Then the centroid $(\bar{x}, \bar{y})$ can be found as
  $$\bar{x} = \frac{M_y}{A} \quad \bar{y} = \frac{M_x}{A}$$
  where A is the area of the region:
  $$A = \int_a^b (f(x) - g(x))\, dx.$$

- Differential Equations

- You should know what it means for a function or equation to be a solution to a given differential equation. You should be able to verify whether or not a given function of equation is a solution to a given differential equation.
- You should be able to solve many **separable** differential equations. A differential equation is separable if it is of the form

  $$\frac{dy}{dx} = f(y)g(x)$$

  and can be potentially solved via integration:

  $$\int \frac{dy}{f(y)} = \int g(x)dx.$$

- You should be able to solve an initial value problem consisting of a separable differential equation and an initial value, or point on the solution curve.
- You should be able to setup and solve a variety of application problems. These will involve a description of a relationship between two quantities (usually, a quantity and time) from which you will need to create and solve a differential equation representation.

  Some categories of this sort of problem include:

  * population modeling, with both exponential and logistic models
  * mixing tank problems
  * orthogonal families of curves