Clarification of the Manipulation of the Differential in Substitution

Suppose the most general antiderivative of $f(x)$ is equal to the most general antiderivative of $g(u)$, where $u = h(x)$.

Let $G(u)$ be the most general antiderivative of $g(u)$. Then

$$\frac{d}{du}G(u) = g(u)$$

and, via the chain rule,

$$\frac{d}{dx}G(u) = \frac{d}{du}G(u) \cdot \frac{du}{dx} = g(u)h'(x) = g(u)\frac{du}{dx} = f(x)$$

With integral notation, our inital assumption was that

$$\int g(u)\,du = \int f(x)\,dx$$

Keep in mind that here we may treat the differential terms as merely place holders indicating with respect to which variable each integral will be integrated.

However, we have just shown that

$$f(x) = g(u)\frac{du}{dx}$$

and so

$$\int \left(g(u)\frac{du}{dx}\right)\,dx = \int f(x)\,dx.$$

Hence, we may equate

$$\int g(u)\frac{du}{dx}\,dx = \int g(u)\,du.$$

Thus it appears that the dx ”cancels”, though is this is not really the case; yet, it is a very useful notational convenience when working with substitutions.