Work via integrations over intervals of time

A cable of length l and linear density δ is hanging vertically down into a mine shaft. The cable has an object attached at the free end of it with weight W. Consider the problem of lifting the cable out of the mine.

One way to solve the problem is to visualize the cable as hanging downward from the origin along the y-axis, so the free end is at $-l$. Then, following our usual procedure, we cut the interval $[-l, 0]$ into n subintervals of equal length Δy. Let y_i be a y value in the i-th subinterval, with $i = 1, ..., n$. Then, the weight of the i-subinterval is

$$\delta \Delta y$$

and the work to lift it to the surface is approximately

$$-y_i \delta \Delta y.$$

Then the work to lift the entire cable to the surface is approximately

$$\sum_{i=1}^{n} -y_i \delta \Delta y$$

Letting n go to infinity, we have

$$\text{work to lift cable only } = \int_{-l}^{0} -y\delta \, dy = \delta \left(\frac{1}{2}y^2\right)\Big|_{-l}^{0} = \frac{1}{2}l^2\delta.$$

The work to lift the object on the cable is simply

$$lW$$

so the total work required to lift the cable and object is

$$lW + \frac{1}{2}l^2\delta.$$

To solve certain variations on this problem, I think is it quite helpful to integrate with respect to *time*. To get an idea of how to do this, let's rework the last example using time.

Suppose we want to lift the cable and object in the above example. Suppose we will use T (say, seconds) to lift the cable, moving it at a constant speed. Then the work takes place in the time interval $[0, T]$. Cut this interval into n equal subintervals of length Δt. Let t_i be a value of t in each subinterval. How much work is done during the i-th subinterval? The cable moves at a speed of

$$\frac{l}{T}$$

so during any subinterval, the cable moved a distance of

$$\frac{l}{T}\Delta t$$

How much force is applied over this distance? The cable starts out with a length of l and is pulled up at a speed of

$$\frac{l}{T}$$

so that after t_i seconds, there is

$$l - \frac{l}{T}t_i$$

units of length of the cable still hanging. This length weighs

$$\delta\left(l - \frac{l}{T}t_i\right)$$

so the force applied during this subinterval is

$$\delta\left(l-\frac{l}{T}t_i\right)+W$$

and the work done during this subinterval is

$$\left(\delta\left(l-\frac{l}{T}t_i\right)+W\right)\frac{l}{T}\Delta T$$

So the total work done is approximately

$$\sum_{i=1}^{n}\left(\delta\left(l-\frac{l}{T}t_i\right)+W\right)\frac{l}{T}\Delta T$$

and the total work done is exactly

$$\int_0^T\left(\delta\left(l-\frac{l}{T}t\right)+W\right)\frac{l}{T}dt$$

This can be simplified:

$$\int_0^T\left(\delta\left(l-\frac{l}{T}t\right)+W\right)\frac{l}{T}dt=\int_0^T\delta\left(l-\frac{l}{T}t\right)\frac{l}{T}\,dt+\int_0^T W\frac{l}{T}\,dt$$
$$=\delta\frac{l}{T}\int_0^T\left(l-\frac{l}{T}t\right)\,dt+W\frac{l}{T}\int_0^T\,dt=\delta\frac{l}{T}\left(lT-\frac{1}{2}lT\right)+Wl$$
$$=\frac{1}{2}\delta l^2+Wl.$$

Note that this is exactly the same as the result we got using the previous method. However, it seems like more of a pain, so why do it? Well, it can be very useful *if we have an object on the cable which has a non-constant weight*. A classic example is **the leaking bucket**. We may suppose that the bucket starts with a weight of W but loses weight at the rate of r units of weight per unit time. So after t units of time, the object will weigh

$$W-rt.$$

To calculate the work now, we simply replace W in the above by this last expression: work is

$$\frac{1}{2}\delta l^2+\int_0^T(W-rt)\frac{l}{T}\,dt=\frac{1}{2}\delta l^2+W\frac{l}{T}\int_0^T\,dt-r\frac{l}{T}\int_0^T t\,dt$$
$$=\frac{1}{2}\delta l^2+Wl-r\frac{l}{T}\frac{1}{2}T^2=\frac{1}{2}\delta l^2+Wl-\frac{1}{2}rlT.$$

For example, suppose you have a 100 ft. cable that weights 5 lbs/ft. connected to a bucket that initially weights 450 pounds, but is losing weight at the rate of 2 lb/sec., and you are going to pull it to the surface is 50 seconds. Then in the above, $l=100$, $W=450$, $\delta=5$, $r=2$, $T=50$ and the work required is

$$\frac{1}{2}(5)(100^2)+450(100)-\frac{1}{2}(2)(100)(50)=65000\text{ft}\cdot\text{lb of work.}$$