## Mass, Centers of Mass, and Double Integrals

Suppose a 2-D region $R$ has density $\rho(x, y)$ at each point $(x, y)$. We can partition $R$ into subrectangles, with $m$ of them in the $x$-direction, and $n$ in the $y$-direction. Suppose each subrectangle has width $\Delta x$ and height $\Delta y$. Then a subrectangle containing the point $(\hat{x}, \hat{y})$ has approximate mass

$$
\rho(\hat{x}, \hat{y}) \Delta x \Delta y
$$

and the mass of $R$ is approximately

$$
\sum_{i=1}^{m} \sum_{j=1}^{n} \rho\left(x_{i}, y_{i}\right) \Delta x \Delta y
$$

where $\left(x_{i}, y_{i}\right)$ is a point in the $i, j$-th subrectangle. Letting $m$ and $n$ go to infinity, we have

$$
M=\operatorname{mass} \text { of } R=\iint_{R} \rho(x, y) d A
$$

Similary, the moment with respect to the $x$ axis can be calculated as

$$
M_{x}=\iint_{R} y \rho(x, y) d A
$$

and the moment with respect to the $y$ axis can be calculated as

$$
M_{y}=\iint_{R} x \rho(x, y) d A
$$

The we may calculate the center of mass of $R$ via

$$
\text { center of mass of } R=(\bar{x}, \bar{y})=\left(\frac{M_{y}}{M}, \frac{M_{x}}{M}\right) .
$$

## Example 1

Let $R$ be the unit square, $R=\{(x, y): 0 \leq x \leq 1,0 \leq y \leq 1\}$. Suppose the density of $R$ is given by the function

$$
\rho(x, y)=\frac{1}{y+1}
$$

so that $R$ is denser near the $x$-axis. As a result, we would expect the center of mass to be below the geometric center, $(1 / 2,1 / 2)$. However, since the density does not depend on $x$, we do expect $\bar{x}=1 / 2$.

We have:

$$
\begin{gathered}
M=\iint_{R} \frac{1}{y+1} d A=\int_{0}^{1} \int_{0}^{1} \frac{1}{y+1} d y d x=\left.\int_{0}^{1} \ln (y+1)\right|_{0} ^{1} d x=\int_{0}^{1} \ln 2 d x=\ln 2=0.693147 \ldots \\
M_{x}=\iint_{R} \frac{y}{y+1} d A=\int_{0}^{1} \int_{0}^{1}\left(1-\frac{1}{y+1}\right) d y d x=\left.\int_{0}^{1}(y-\ln (y+1))\right|_{0} ^{1} d x \\
=\int_{0}^{1}(1-\ln 2) d x=1-\ln 2=0.306852819 \ldots \\
M_{y}=\iint_{R} \frac{x}{y+1} d A=\int_{0}^{1} \int_{0}^{1} \frac{x}{y+1} d y d x=\int_{0}^{1} x \ln 2 d x=\left.\frac{1}{2} x^{2} \ln 2\right|_{0} ^{1}=\frac{1}{2} \ln 2=0.346573590 \ldots
\end{gathered}
$$

Thus the center of mass is

$$
(\bar{x}, \bar{y})=\left(\frac{M_{y}}{M}, \frac{M_{x}}{M}\right)=\left(\frac{\frac{1}{2} \ln 2}{\ln 2}, \frac{1-\ln 2}{\ln 2}\right)=\left(\frac{1}{2}, 0.442095 \ldots\right) .
$$

## Example 2 (Polar)

Let $0 \leq z \leq 1$. Let $R$ be the polar region

$$
R=\left\{(r, \theta): z \leq r \leq 1,0 \leq \theta \leq \frac{\pi}{2}\right\}
$$

Suppose $R$ has constant density $\rho$. Then:

$$
\begin{gathered}
M=\iint_{R} \rho d A=\rho \iint_{R} d A=\rho \cdot \text { area of } R=\rho\left(\frac{\pi}{4}-\frac{\pi z^{2}}{4}\right)=\frac{\pi}{4} \rho\left(1-z^{2}\right) . \\
M_{x}=\iint_{R} \rho y d A=\rho \int_{z}^{1} \int_{0}^{\pi / 4} r^{2} \sin \theta d \theta d r=\rho \int_{z}^{1}-\left.r^{2} \cos \theta\right|_{0} ^{\pi / 2} d r=\rho \int_{z}^{1} r^{2} d r=\frac{1}{3} \rho\left(1-z^{3}\right) . \\
M_{y}=\iint_{R} \rho x d A=\rho \int_{z}^{1} \int_{0}^{\pi / 2} r^{2} \cos \theta d \theta d r=\left.\rho \int_{z}^{1} r^{2} \sin \theta\right|_{0} ^{\pi / 2} d r=\rho \int_{z}^{1} r^{2} d r=\frac{1}{3} \rho\left(1-z^{3}\right) .
\end{gathered}
$$

Thus, the center of mass is

$$
(\bar{x}, \bar{y})=\left(\frac{\frac{1}{3}\left(1-z^{3}\right)}{\frac{\pi}{4}\left(1-z^{2}\right)}, \frac{\frac{1}{3}\left(1-z^{3}\right)}{\frac{\pi}{4}\left(1-z^{2}\right)}\right) .
$$

An interesting feature of this region is that if $z$ is sufficiently large, the center of mass will be outside the region. This happens when the distance from the center of mass to $(0,0)$ is less than $z$. That is, when

$$
\sqrt{2} \frac{\frac{1}{3}\left(1-z^{3}\right)}{\frac{\pi}{4}\left(1-z^{2}\right)}<z
$$

By factoring, we see that this is equivalent to

$$
\frac{\frac{\sqrt{2}}{3}\left(1+z+z^{2}\right)}{\frac{\pi}{4}(1+z)}<z
$$

The critical $z$ value is the positive solution to

$$
0=z^{2}+z-\frac{\frac{\sqrt{2}}{3}}{\frac{\pi}{4}-\frac{\sqrt{2}}{3}}
$$

which is about $0.82337397 \ldots$ Thus, if $z>0.82337397 \ldots$, the region is very thin, and the center of mass lies outside of the region.

