Math 126 C, D - Spring 2006
Mid-Term Exam Number Two
Hints and Answers

1. Version 1

   Find the slope of the tangent line to the polar curve

   $$r = \frac{1}{\theta}, \theta > 0$$

   at the point where it intersects the cartesian curve

   $$x^2 + y^2 = \frac{1}{9}.$$

   The intersection will occur when $r = \frac{1}{3}$, i.e., when $\theta = 3$. The slope of the tangent line is

   $$\frac{dy}{dx} = \frac{-\frac{1}{9}\sin 3 + \frac{1}{3}\cos 3}{-\frac{1}{9}\cos 3 - \frac{1}{3}\sin 3}$$

   Version 2

   Find the slope of the tangent line to the polar curve

   $$r = \frac{1}{\theta}, \theta > 0$$

   at the point where it intersects the cartesian curve

   $$x^2 + y^2 = \frac{1}{4}.$$

   The intersection will occur when $r = \frac{1}{2}$, i.e., when $\theta = 2$. The slope of the tangent line is

   $$\frac{dy}{dx} = \frac{-\frac{1}{4}\sin 2 + \frac{1}{2}\cos 2}{-\frac{1}{4}\cos 2 - \frac{1}{2}\sin 2}$$

2. Version 1

   At what point(s) is the tangent line to the curve

   $$x = t^3 - 3t, y = t^2 + 2t$$

   parallel to the line with parametric equations

   $$x = 3t + 5, y = t - 6 \text{ ?}$$

   A sufficient condition is that

   $$t^2 - 2t - 3 = (t - 3)(t + 1) = 0$$

   but $t = -1$ is not a solution: $t = 3$ is the only point.

   Version 2

At what point(s) is the tangent line to the curve

$$x = \frac{2}{3}t^3 + 2t, y = 3t^2 - 7t$$

parallel to the line with parametric equations

$$x = 2t - 3, y = t + 5 \text{ ?}$$

A sufficient condition is that

$$0 = t^2 - 6t + 8 = (t - 4)(t - 2)$$

and both $t = 4$ and $t = 2$ are solutions.

3. Version 1

   For any $m > 0$, the helix determined by the position function

   $$\vec{r}(t) = \langle \cos t, \sin t, mt \rangle$$

   has constant curvature that depends on m. Find the value of m such that the radius of curvature at any point on the curve is 3.

   The curvature at any point on the curve is $\frac{1}{m^2+1}$, so $m = \sqrt{2}$.

   Version 2

   For any $m > 0$, the helix determined by the position function

   $$\vec{r}(t) = \langle \cos t, \sin t, mt \rangle$$

   has constant curvature that depends on m. Find the value of m such that the radius of curvature at any point on the curve is 6.

   The curvature at any point on the curve is $\frac{1}{m^2+1}$, so $m = \sqrt{5}$.

4. Version 1

   A particle is moving so that its position is given by the vector function

   $$\vec{r}(t) = \langle t^2, t, 5t \rangle$$

   Find the tangent and normal components of the particle's acceleration vector.

   The tangent component is

   $$a_T = \frac{\vec{r}'(t) \cdot \vec{r}''(t)}{|\vec{r}(t)|} = \frac{4t}{\sqrt{4t^2 + 26}}$$

   and the normal component is

   $$a_N = \frac{|\vec{r}'(t) \times \vec{r}'(t)|}{|\vec{r}(t)|} = \frac{\sqrt{104}}{\sqrt{4t^2 + 26}}$$

   Version 2

   A particle is moving so that its position is given by the vector function

   $$\vec{r}(t) = \langle 3t, t^2, t \rangle$$

   Find the tangent and normal components of the particle's acceleration vector.

The tangent component is

$$a_T = \frac{\vec{r}'(t) \cdot \vec{r}''(t)}{|\vec{r}(t)|} = \frac{4t}{\sqrt{4t^2 + 10}}$$

and the normal component is

$$a_N = \frac{|\vec{r}'(t) \times \vec{r}'(t)|}{|\vec{r}(t)|} = \frac{\sqrt{40}}{\sqrt{4t^2 + 10}}$$

5. Version 1

Reparametrize the curve

$$\vec{r}(t) = \langle 5t - 1, 2t, 3t + 2 \rangle$$

with respect to arc length measured from the point where $t = 0$ in the direction of increasing t.

$$\vec{r}(s) = \langle \frac{5}{\sqrt{38}} s - 1, \frac{2}{\sqrt{38}} s, \frac{3}{\sqrt{38}} s + 2 \rangle$$

Version 2

Reparametrize the curve

$$\vec{r}(t) = \langle 2t, 7t + 5, 4t - 6 \rangle$$

with respect to arc length measured from the point where $t = 0$ in the direction of increasing t.

$$\vec{r}(s) = \langle \frac{2}{\sqrt{69}} s, \frac{7}{\sqrt{69}} s + 5, \frac{4}{\sqrt{69}} s - 6 \rangle$$

6. Version 1

Let $f(x, y) = x^2 y + x \sin y - \ln(x - y^2)$.

   (a) Find $f_y(x, y)$.
   (b) Find $f_{xy}(x, y)$.

(a) $f_y(x, y) = x^2 + x \cos y + \frac{2y}{x - y^2}$

(b) $f_{xy}(x, y) = 2x + \cos y - \frac{2y}{(x - y^2)^2}$

Version 2

Let $f(x, y) = x^2 y + x \cos y - \ln(x - y^2)$.

   (a) Find $f_y(x, y)$.
   (b) Find $f_{xy}(x, y)$.

(a) $f_y(x, y) = x^2 - x \sin y + \frac{2y}{x - y^2}$

(b) $f_{xy}(x, y) = 2x - \sin y - \frac{2y}{(x - y^2)^2}$