Math 126D - Spring 2002
Second Mid-Term Exam Solutions
May 23, 2002

1. (10 points) Indicate whether each of the following statements is true or false by circling T or F.

   T **F** If $\vec{a} \cdot \vec{b} = 0$, then $\vec{a}$ and $\vec{b}$ are parallel.

   T F $\langle a, 0, 0\rangle \times \langle 0, b, 0\rangle = \langle 0, 0, ab\rangle$.

   T F If $ab = 1$, then $\langle a, 1, 1\rangle \times \langle 1, b, 1\rangle = \langle 1 - b, 1 - a, 0\rangle$.

   T **F** If $\vec{a} \cdot \vec{b} = 0$ then either $\vec{a}$ or $\vec{b}$ is the zero vector.

   T F A plane has an infinite number of vectors which are perpendicular to it.

2. (10 points) Determine the Taylor polynomial of degree 3 for the function $f(x) = (2x + 1)^{2/3}$ centered at $a = 0$.

   We have:

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$(2x + 1)^{2/3}$	1
1	$\frac{2}{3}(2x + 1)^{-1/3}(2)$	$\frac{4}{3}$
2	$-\frac{2}{9}(2x + 1)^{-4/3}(2^2)$	$-\frac{8}{9}$
3	$\frac{8}{27}(2x + 1)^{-7/3}(2^3)$	$\frac{64}{27}$

   Thus, the Taylor polynomial of degree 3 for $f(x) = (2x + 1)^{2/3}$ is

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 = 1 + \frac{4}{3}x - \frac{4}{9}x^2 + \frac{32}{81}x^3.$$

3. (10 points) Find the equation of the plane containing the line

$$x = 3 + t, y = 4 - 2t, z = 1 - t$$

   and the point $(1, 2, 5)$.

   The direction vector of the line is $\langle 1, -2, -1\rangle$. Call this vector $\vec{a}$. Letting $t = 0$, we find a point on the line: $x = 3, y = 4, z = 1$, i.e., the point $(3, 4, 1)$. The vector pointing from $(1, 2, 5)$ to the point $(3, 4, 1)$ is the vector $\langle 2, 2, -4\rangle$. Call this vector $\vec{b}$. Vectors $\vec{a}$ and $\vec{b}$ are both parallel to the plane we're looking for. So, the cross product of $\vec{a}$ and $\vec{b}$ is parallel to the normal vector of this plane. We find $\vec{a} \times \vec{b} = \langle 10, 2, 6\rangle$, and the equation of the plane is

$$10(x - 1) + 2(y - 2) + 6(z - 5) = 0$$

or

$$10x + 2y + 6z - 44 = 0.$$

4. (10 points) Consider the space curve defined by

$$\vec{r}(t) = \langle 4t^3 + 12t^2,\ 4t^3 - 6t^2,\ 3t^4 - 18t^2 \rangle.$$

Find all values of t such that the tangent vector to $\vec{r}(t)$ is parallel to the line

$$x = 5 - 2t,\ y = 8 + 4t,\ z = -7 - 4t.$$

The direction vector of the line is $\langle -2, 4, -4 \rangle$. We have

$$\vec{r}'(t) = \langle 12t^2 + 24t,\ 12t^2 - 12t,\ 12t^3 - 36t \rangle.$$

So, we want to find t and c so that

$$\vec{r}'(t) = c\langle -2, 4, -4 \rangle.$$

At such a t, we must have

$$12t^2 - 12t = -2(12t^2 + 24t)$$

from which we can conclude that $t = 0$ or $t = -1$.

However, $\vec{r}'(0)$ is the zero vector, which is parallel to nothing, and $\vec{r}'(1) = \langle -12, 24, 24 \rangle$ which is clearly not parallel to $\langle -2, 4, -4 \rangle$.

There are no values of t at which the tangent vector is parallel to the given line.

5. (10 points) Consider the curve defined by the parametric equations

$$x = t^3 + 5t,\ y = t^3 - 12t.$$

(a) Find all points on the curve where $\dfrac{dy}{dx} = 0$.

We have

$$\frac{dy}{dt} = 3t^2 - 12,\ \frac{dx}{dt} = 3t^2 + 5$$

so that

$$\frac{dy}{dx} = \frac{3t^2 - 12}{3t^2 + 5}.$$

Note that $3t^2 + 5 \neq 0$ for all t, so that if $\dfrac{dy}{dx} = 0$, then $3t^2 - 12 = 0$, i.e. $t = 2$ or $t = -2$.

(b) Find the concavity $\frac{d^2y}{dx^2}$ at each of the points you found in part (a).

We have, via the chain rule,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}.$$

Now,

$$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{102t}{3t^2+5}$$

so

$$\frac{d^2y}{dx^2} = \frac{102t}{(3t^2+5)^2}.$$

Thus, at $t = 2$, $\frac{d^2y}{dx^2} = \frac{12}{17}$, and at $t = -2$, $\frac{d^2y}{dx^2} = \frac{-12}{17}$.

6. (10 points) Find the equation of the line that is the intersection of the plane

$$x + y + 3z = 8$$

with the plane

$$2x - y - z = 4.$$

There are a few ways to solve this. Here's one.

If we have two points on the line, then we can determine the equation of the line. Algebraically, the equations

$$x + y + 3z = 8 \text{ and } 2x - y - z = 4$$

can be solved by choosing arbitrary values of x, say. Let $x = 0$. Then we have

$$y + 3z = 8, \; -y - z = 4$$

which, by adding, yields

$$2z = 12$$

or $z = 6$. From $-y - z = 4$, we have $y = -10$. Hence the point $(0, -10, 6)$ is on the line. Letting $x = 2$, we get the point $(2, -3, 3)$. Hence a direction vector for the line is $\langle 2 - 0, -3 - (-10), 3 - 6\rangle = \langle 2, 7, -3\rangle$, so a set of equations for the line is

$$x = 2t, \; y = -10 + 7t, \; z = 6 - 3t.$$