Tangent Spirals

When studying curves in the plane defined via polar coordinates, one class of interest is curves defined by equations of the form

$$r = f(\theta).$$

This equation can be considered analogous to the cartesian equation $y = f(x)$.

Now, if f is a monotonic function (i.e., it is always increasing, or always decreasing), then the curve defined by $r = f(\theta)$ is generally called a *spiral*. The spiral $r = \theta$ is the simplest example.

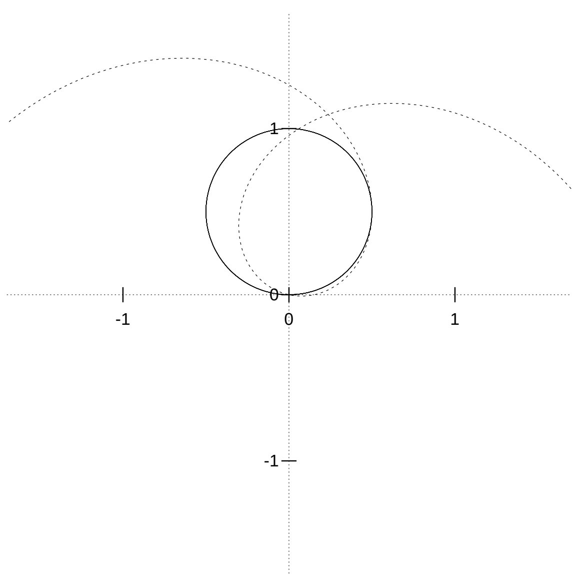
In studying curves of the form $y = f(x)$, the tangent line is a useful tool. We can consider the tangent line to a polar curve $r = f(\theta)$; however, for a curve of this sort, $\frac{dy}{dx}$ is generally a messy affair. Instead, let us make an analogy, as suggested by the following table.

	Cartesian	polar
curve defined by	$y = f(x)$	$r = f(\theta)$
derivative	$\frac{dy}{dx} = f'(x)$	$\frac{dr}{d\theta} = f'(\theta)$
tangent curve	$y = f'(x_0)(x - x_0) + f(x_0)$	$r = f'(\theta_0)(\theta - \theta_0) + f(x_0)$
tangent curve type	line	spiral

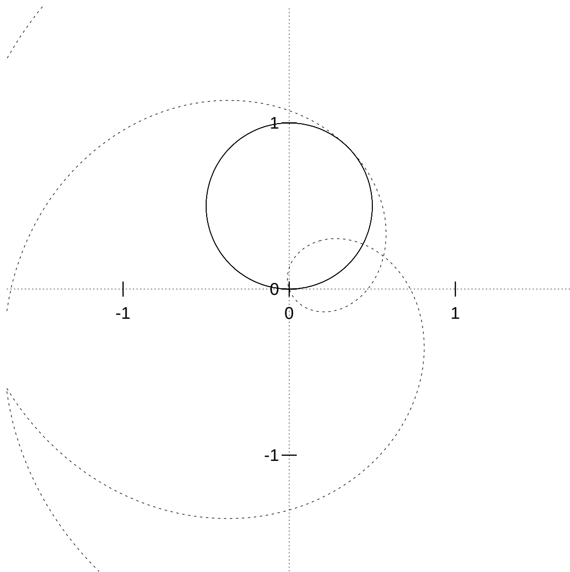
In this way, we can define the *tangent spiral* to a curve at a point.

Here are some graphic examples.

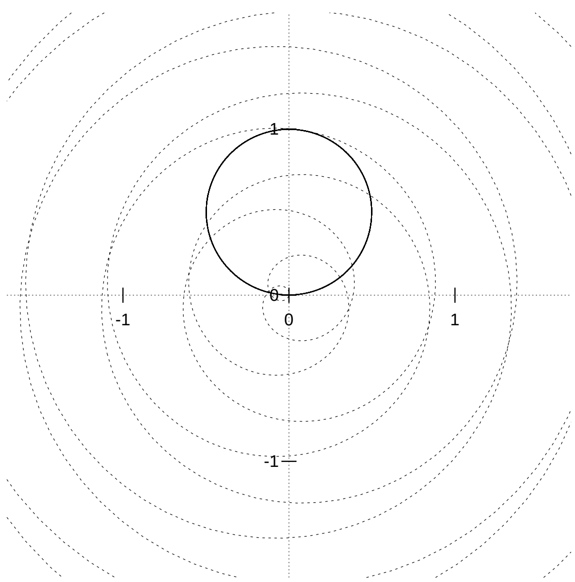
First, is the circle $r = \sin\theta$ together with its tangent spiral at $\theta = \dfrac{\pi}{4}$.



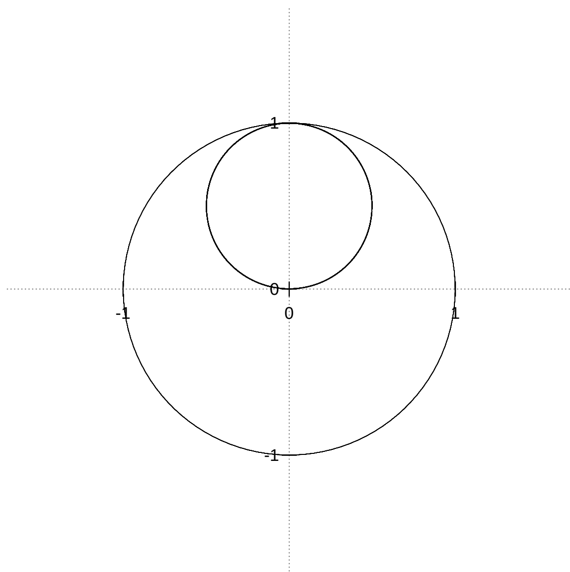
Next, the same circle with its tangent spiral at $\theta = \dfrac{3\pi}{8}$.



An interesting phenomenon occurs as we approach the point at $\theta = \frac{\pi}{2}$. Here's the picture for $\theta = \frac{9}{10}\frac{\pi}{2}$:



When we get to $\theta = \frac{\pi}{2}$, the result looks like this:



This is a circle, not a spiral. What happened? Well, if

$$r = \sin\theta$$

then

$$\frac{dr}{d\theta} = \cos\theta = \cos\frac{\pi}{2} = 0$$

when $\theta = \frac{\pi}{2}$. So, our formula for the tangent spiral gives us

$$r = f'(\theta_0)(\theta - \theta_0) + f(\theta_0) = 0 + \sin\frac{\pi}{2} = 1.$$

We see that whenever $f'(\theta_0)$ is zero, our formula for the tangent spiral becomes the formula for a circle. Note that if $f'(\theta) = 0$, then the would-be spiral has equation $r =$ a constant, not a monotonic function of θ.

This is analogous to the horizontal tangent which occurs when $f'(x) = 0$.

A natural question at this point: is the "tangent spiral" actually tangent to the curve? That is, is the tangent line to the curve the same as the tangent line to the spiral? Let's see.

For a curve defined by $r = f(\theta)$, we have the slope of the tangent line at the point $\theta = \theta_0$ is

$$\frac{dy}{dx} = \frac{f'(\theta_0)\sin\theta_0 + f(\theta_0)\cos\theta_0}{f'(\theta_0)\cos\theta_0 - f(\theta_0)\sin\theta_0}.$$

The spiral "tangent" to the curve at $\theta = \theta_0$ has equation

$$r = f'(\theta_0)(\theta - \theta_0) + f(\theta_0) = g(\theta), \text{ say.}$$

Then, the slope of the tangent line to the spiral at this point is

$$\frac{g'(\theta_0)\sin\theta_0 + g(\theta_0)\cos\theta_0}{g'(\theta_0)\cos\theta_0 - g(\theta_0)\sin\theta_0} = \frac{f'(\theta_0)\sin\theta_0 + (f'(\theta_0)(\theta_0 - \theta_0) + f(\theta_0))\cos\theta_0}{f'(\theta_0)\cos\theta - (f'(\theta_0)(\theta_0 - \theta_0) + f(\theta_0))\sin\theta_0}$$

and this simplfies precisely to $\frac{dy}{dx}$ above.

So the tangent spirals are tangent.