Taylor polynomials and finding intervals with required error bounds

Here are two examples of finding intervals on which required error bounds between a function and T_2 are attained.

1. Let's consider the function $f(x) = x\sin^3 x$. Suppose we take $b = 0$, and want an interval around $b = 0$ such that the error
$$|f(x) - T_2(x)|$$
is less than 0.01.

   We being by finding $f'''(x)$. One can show that
$$f'''(x) = -9\sin^3(x) - 21x\cos(x)\sin^2(x) + 18\cos^2(x)\sin(x) + 6x\cos^3(x)\,.$$

   To find a value of M so that
$$|f'''(x)| \le M$$
on some interval, we begin by choosing an interval, fairly arbitrarily.

   Let's suppose $-1 \le x \le 1$. Then, using the fact that $|\sin x| \le 1$ and $|\cos x| \le 1$ and
$$|A + B| \le |A| + |B|$$
for any A and B, we may conclude that
$$|f'''(x)| \le 9 + 21 + 18 + 6 = 54.$$

   Hence, for x in $-1 \le x \le 1$, we have $M = 54$ and so, by our error bound for T_2,
$$|\text{error}| \le \frac{M}{6}|x - 0|^3 = 9|x|^3.$$

   Now, we want the error to be less than 0.01, and we can achieve that if
$$9|x|^3 \le 0.01$$
which solves to
$$|x| \le 0.10357.$$

   Note that this is inside the assumed interval $-1 \le x \le 1$.

   Thus the error is less than 0.01 for $-0.10356 \le x \le 0.10356$.

   We could stop here, since we've answered the question.

   However, we can improve this interval (i.e., make it bigger) by choosing a different starting interval. Suppose now that $-0.5 \le x \le 0.5$. Then
$$|f'''(x)| \le 9 + 21(0.5) + 18 + 6(0.5) = 40.5$$
and our error satisfies
$$|\text{error}| \le \frac{40.5}{6}|x|^3$$
so solving
$$\frac{40.5}{6}|x|^3 \le 0.01$$
we find the error is less than 0.01 for $-0.1139983 \le x \le 0.1139983$. This is a tiny improvement, so hardly worth the trouble.

2. Suppose $f(x) = \sqrt{x}$ and we want T_2 for f based at $b = 1$. Suppose we want an error of less than 0.01. We begin by finding some derivatives. In particular, we need

$$f'''(x) = \frac{15}{8}x^{-7/2}$$

in order use Taylor's error bound to bound the error.

One approach is to begin with an interval around b, and bound $f'''(x)$ on this interval to get a value of M. Then, use the error bound to solve for a smaller interval on which we can achieve the required error.

For instance, consider the interval $[0.5, 1.5]$. We see that $f'''(x)$ is strictly decreasing on this interval, so it is largest at the left endpoint, so we may take

$$M = f'''(0.5) = 21.2132....$$

Then, the error bound says

$$|\text{error}| \leq \frac{21.2133}{6}|x-1|^3$$

and since we want this less than 0.01, we solve

$$\frac{21.2133}{6}|x-1|^3 \leq 0.01$$

to find

$$|x-1| \leq 0.14142,$$

that is, [0.85858,1.14142] is an interval on which we can know the error is less than 0.01.

We do need to note that this interval is inside the assumed interval [0.5,1.5]. Had it not been, we would have concluded that the required error was achieved on the entire interval [0.5,1.5].

For instance, suppose we started with the interval $[0.8, 1.2]$. On this interval,

$$|f'''(x)| \leq f'''(0.8) = 4.09436....$$

Then we want

$$|\text{error}| \leq \frac{4.0944}{6}|x-1|^3 < 0.01$$

which yields

$$|x-1| \leq 0.24471109514....$$

Now, this says that the error is less than 0.01 on the interval $[0.7553, 1.2447]$ provided the bound for M is valid on this interval: but it is not. Hence, what we *can* say is that the error is less than 0.01 on the largest subinterval of $[0.7553, 1.2447]$ for which our bound for M holds. That largest subinterval is the interval we initially assumed to get that bound, $[0.8, 1.2]$.

In this way, we make a slight improvement over our last interval.