Here are two examples of finding intervals on which required error bounds between a function and T_2 are attained.

1. Let's consider the function $f(x) = e^{\sin x}$.

   Suppose we are interested in approximating this function within an error of, say, 0.01 in an interval around $b = 0$. Let's do this with a second-degree Taylor polynomial.

   We begin by getting the necessary derivatives:

   $$f'(x) = e^{\sin x} \cos x$$

   $$f''(x) = e^{\sin x} \left(\cos^2 x - \sin x\right)$$

   and

   $$f'''(x) = e^{\sin x} \left(\cos^3 x - 3 \cos x \sin x - \cos x\right).$$

   Now, Taylor's error bound says that

   $$|f(x) - T_2(x)| \leq \frac{M}{6}|x|^3$$

   for x in an interval I containing the base b for which

   $$|f'''(x)| \leq M$$

   everywhere in I.

   What interval will get this error bound below 0.01?

   We start by considering $f'''(x)$, and seeing what bounds on it are possible. We can use the fact that, for any x, $e^{\sin x} < e$, and that

   $$|\cos^3 x - 3 \cos x \sin x - \cos x| \leq 1 + 3 + 1 = 5$$

   to get a value of $M = 5e$ valid for all x.

   Then our error bound, and our goal of getting the error below 0.01, yields

   $$|\text{error}| \leq \frac{5e}{6}|x|^3 \leq 0.01$$

   which results in

   $$x \leq 0.164044719...$$

   Thus, in the interval $[-0.164, 0.164]$, we can be certain that the error $|f(x) - T_2(x)|$ is less than 0.01. (Actually, the error in this interval is *much* smaller than 0.01 (as you can check numerical or by using software to graph $f(x)$ and $T_2(x)$), but it would take more work to reduce our current bound.

   (For an improvement, it is not terribly hard to show that

   $$|\cos^3 x - \cos x| \leq 0.5 \text{ for all } x$$

   and

   $$|\cos x \sin x| \leq 0.5 \text{ for all } x$$

   so our M value of $5e$ can be replaced by $e(0.5 + 1.5) = 2e$. This would better the interval we found to [-0.2095,0.2095].)

2. Suppose $f(x) = \sqrt{x}$ and we want T_2 for f based at $b = 1$. Suppose we want an error of less than 0.01. We begin by finding some derivatives. In particular, we need

$$f'''(x) = \frac{15}{8}x^{-7/2}$$

in order use Taylor's error bound to bound the error.

One approach is to begin with an interval around b, and bound $f'''(x)$ on this interval to get a value of M. Then, use the error bound to solve for a smaller interval on which we can achieve the required error.

For instance, consider the interval $[0.5, 1.5]$. We see that $f'''(x)$ is strictly decreasing on this interval, so it is largest at the left endpoint, so we may take

$$M = f'''(0.5) = 21.2132....$$

Then, the error bound says

$$|\text{error}| \leq \frac{21.2133}{6}|x-1|^3$$

and since we want this less than 0.01, we solve

$$\frac{21.2133}{6}|x-1|^3 \leq 0.01$$

to find

$$|x-1| \leq 0.14142,$$

that is, [0.85858,1.14142] is an interval on which we can know the error is less than 0.01.

We do need to note that this interval is inside the assumed interval [0.5,1.5]. Had it not been, we would have concluded that the required error was achieved on the entire interval [0.5,1.5].

For instance, suppose we started with the interval $[0.8, 1.2]$. On this interval,

$$|f'''(x)| \leq f'''(0.8) = 4.09436....$$

Then we want

$$|\text{error}| \leq \frac{4.0944}{6}|x-1|^3 < 0.01$$

which yields

$$|x-1| \leq 0.24471109514....$$

Now, this says that the error is less than 0.01 on the interval $[0.7553, 1.2447]$ provided the bound for M is valid on this interval: but it is not. Hence, what we *can* say is that the error is less than 0.01 on the largest subinterval of $[0.7553, 1.2447]$ for which our bound for M holds. That largest subinterval is the interval we initially assumed to get that bound, $[0.8, 1.2]$.

In this way, we make a slight improvement over our last interval.