Why does $\dfrac{x^n}{n!}$ go to zero as n goes to infinity, for all values of x?

As n goes to infinity, x stays fixed. The basic idea is that for very large n, the denominator has lots of factors which are larger than x, and this makes the quotient small.

Let $\hat{x}$ be an integer larger than x. For instance, if $x = 13.23123$, we might take $\hat{x} = 14$, but anything larger would work, too.

Suppose that $n > 2\hat{x}$. Then

$$\frac{\hat{x}}{n} < \frac{1}{2},$$

and we have

$$\frac{x^n}{n!} < \frac{\hat{x}^n}{n!} = \left(\frac{\hat{x}}{1}\frac{\hat{x}}{2}\cdots\frac{\hat{x}}{2\hat{x}}\right)\left(\left(\frac{\hat{x}}{2\hat{x}+1}\right)\left(\frac{\hat{x}}{2\hat{x}+2}\right)\cdots\left(\frac{\hat{x}}{n}\right)\right) < k\left(\frac{1}{2}\right)^{n-2\hat{x}}$$

where

$$k = \left(\frac{\hat{x}}{1}\frac{\hat{x}}{2}\cdots\frac{\hat{x}}{2\hat{x}}\right).$$

Notice that k does not depend on n, but only on x, so it is constant as n varies. Hence, we can conclude that

$$\lim_{n\to\infty}\frac{x^n}{n!} \le \lim_{n\to\infty} k\left(\frac{1}{2}\right)^{n-2\hat{x}} = \lim_{n\to\infty} k2^{2\hat{x}}\left(\frac{1}{2}\right)^n = k2^{2\hat{x}}\lim_{n\to\infty}\left(\frac{1}{2}\right)^n = 0$$

and so

$$\lim_{n\to\infty}\frac{x^n}{n!} = 0.$$

Here are some illustrating calculations.

Suppose $x = 11$. Then we have the following table.

n	$\dfrac{11^n}{n!}$
1	11.000000
2	60.500000
3	221.83333
4	610.04166
5	1342.0916
6	2460.5013
7	3866.5021
8	5316.4405
9	6497.8717
10	7147.6589
11	7147.6589
12	6552.0206
13	5544.0174
14	4356.0137
15	3194.4100
16	2196.1569
17	1421.0427
18	868.41499
19	502.76657
20	276.52161
21	144.84465
22	72.422328
23	34.636765
24	15.875184
25	6.9850810
26	2.9552266
27	1.2039812
28	0.472992617
29	0.179410993
30	0.065784030
35	0.000271963
40	0.000000554

We see that, at first $\dfrac{x^n}{n!}$ grows, since $n < 11$. Notice that the growth stops at $n = 10$; in fact,

$$\frac{11^{11}}{11!} = \frac{11^{10}}{10!}.$$

After that, every new factor added to the denominator is greater than 11, and so we multiply each value in the right column by a factor less than one. Plus, these factors get smaller and smaller, so the right hand column shrinks faster and faster as n increases. Notice that after $n = 22$, the value is more than cut in half at each step. The result is that, by $n = 40$, the value is quite small, and this supports our argument above that the value will approach zero as n tends to infinity.