Math 300 D - Autumn 2014
Final Exam
December 9, 2014
Solutions

1. *Let $\mathcal{F}$ and $\mathcal{G}$ be families of sets. Prove that $(\cap\mathcal{F}) \cap (\cap\mathcal{G}) = \cap(\mathcal{F} \cup \mathcal{G})$.*

   Proof: Let $\mathcal{F}$ and $\mathcal{G}$ be families of sets.

   Suppose $x \in \cap\mathcal{F} \cap \cap\mathcal{G}$.

   Then $x \in \cap\mathcal{F}$ and $x \in \cap\mathcal{G}$.

   Suppose $M \in \mathcal{F}$. Then $x \in M$.

   Suppose $N \in \mathcal{G}$. Then $x \in N$.

   Suppose $P \in \mathcal{F} \cup \mathcal{G}$. Then $P \in \mathcal{F}$ or $P \in \mathcal{G}$. Hence, $x \in P$.

   Thus, x is an element of every set in $\mathcal{F} \cup \mathcal{G}$, so $x \in \cap(\mathcal{F} \cup \mathcal{G})$.

   Hence, $(\cap\mathcal{F}) \cap (\cap\mathcal{G}) \subseteq \cap(\mathcal{F} \cup \mathcal{G})$.

   Now, suppose $x \in \cap(\mathcal{F} \cup \mathcal{G})$.

   Suppose $M \in \mathcal{F} \cup \mathcal{G}$. Then $x \in M$.

   Suppose $N \in \mathcal{F}$. Then $N \in \mathcal{F} \cup \mathcal{G}$, and so $x \in N$.

   Hence, x is in every set in $\mathcal{F}$, and so $x \in \cap\mathcal{F}$.

   Suppose $P \in \mathcal{G}$. Then $P \in \mathcal{F} \cup \mathcal{G}$, and so $x \in P$.

   Hence, x is in every set in $\mathcal{G}$, and so $x \in \cap\mathcal{G}$.

   Thus, $x \in (\cap\mathcal{F}) \cap (\cap\mathcal{G})$.

   Therefore $\cap(\mathcal{F} \cup \mathcal{G}) \subseteq (\cap\mathcal{F}) \cap (\cap\mathcal{G})$, and so $(\cap\mathcal{F}) \cap (\cap\mathcal{G}) = \cap(\mathcal{F} \cup \mathcal{G})$. ■

2. *Let a and b be integers. Prove that $a(b + a + 1)$ is odd iff a and b are both odd.*

   Proof: Let a and b be integers.

   Suppose a is even.

   Then $a = 2k$ for some $k \in \mathbb{Z}$.

   Then $a(b+a+1) = 2k(b+a+1)$, and since a, b, and k are integers, $k(b+a+1)$ is an integer.

   Hence, $2 \mid a(b + a + 1)$, i.e., $a(b + a + 1)$ is even.

   Suppose a is odd and b is even.

   Then $a = 2k + 1$ and $b = 2m$ for some integers k and m.

   Then $a(b + a + 1) = a(2k + 2m + 1 + 1) = a(2k + 2m + 2) = 2a(k + m + 1)$. Since a, k, and m are integers, $a(k + m + 1)$ is an integer, and so $2 \mid a(b + a + 1)$, i.e., $a(b + a + 1$ is even.

   Suppose a is odd and b is odd.

Then $a = 2k+1$ and $b = 2m+1$ for some integers k and m.

Then

$$\begin{aligned} a(b+a+1) &= (2k+1)(2k+2m+3) \\ &= 4k^2 + 4mk + 8k + 2m + 3 \\ &= 2(2k^2 + 4mk + 4k + m + 1) + 1. \end{aligned}$$

Since k and m are integers, $2k^2 + 4mk + 4k + m + 1$ is an integer, and so $a(b+a+1)$ is odd.

Thus, $a(b+a+1)$ is odd iff a and b are both odd.■

3. *Let $A, B, C,$ and D be sets. Suppose $A \cap C = B \cap D = \varnothing$. Suppose $A \sim B$ and $C \sim D$. Prove that $A \cup C \sim B \cup D$.*

   Proof: Let $A, B, C,$ and D be sets. Suppose $A \cap C = B \cap D = \varnothing$. Suppose $A \sim B$ and $C \sim D$. Then there exist bijections $f : A \to B$ and $g : C \to D$.

   Define a function $h : A \cup C \to B \cup D$ by

$$h(x) = \begin{cases} f(x) & \text{if } x \in A \\ g(x) & \text{if } x \in C \end{cases}$$

   Since $A \cap C = \varnothing$, the function is well-defined: given an x in $A \cup C$, x is either in A or C, but not both.

   Suppose $h(x_1) = h(x_2)$ for some x_1 and x_2 in $A \cup C$.

   Suppose $x_1 \in A$, and $x_2 \in A$. Then $h(x_1) = f(x_1) = h(x_2) = f(x_2)$, and so $x_1 = x_2$ since f is injective.

   Suppose $x_1 \in C$, and $x_2 \in C$. Then $h(x_1) = g(x_1) = h(x_2) = g(x_2)$, and so $x_1 = x_2$ since g is injective.

   Suppose x_1 and x_2 are not both in A or both in C. Without loss of generality, suppose $x_1 \in A$ and $x_2 \in C$. Then $h(x_1) = h(x_2)$ implies that $f(x_1) = g(x_2)$. But $f(x_1) \in B$ and $g(x_2) \in D$, so $B \cap D \neq \varnothing$. This is a contradiction to our assumption that $B \cap D = \varnothing$. Hence, x_1 and x_2 must both be in A or both be in C.

   Thus, $x_1 = x_2$, and so h is one-to-one.

   Suppose $r \in B \cup D$.

   Then $r \in B$ or $r \in D$.

   Suppose $r \in B$. Then, since f is surjective, there exists an $a \in A$ such that $f(a) = r$. Also, $a \in A \cup C$, and $h(a) = r$.

   Suppose $r \in D$. Then, since g is surjective, there exists a $c \in C$ such that $g(c) = r$. Also, $c \in A \cup C$ and $h(c) = r$.

   Hence, there exists $x \in A \cup B$ with $h(x) = r$, so h is surjective.

   Thus, h is a bijection, and so $A \cup C \sim B \cup D$. ■

4. (a) *Give an example of a function $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ such that f is one-to-one, but not onto (i.e., f is injective but not surjective). Prove that f is one-to-one and not onto.*

Note: There are many examples you could give. If you are using this exam solution to study, I recommend creating a few, as different from each other as possible.

One example is $f(x) = x + 1$.

We note that since $x > 0$, $x + 1 > 1$ and so there is no a such that $f(a) = 1$. So f is not onto.

Suppose $f(x_1) = f(x_2)$. Then $x_1 + 1 = x_2 + 1$, and so $x_1 = x_2$. So f is one-to-one. ■

(b) *Give an example of a function $g : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ such that g is onto, but not one-to-one (i.e., g is surjective, but not injective). Prove that g is onto and not one-to-one.*

Note: There are many examples you could give. If you are using this exam solution to study, I recommend creating a few, as different from each other as possible.

Here is one example. Let

$$g(x) = \begin{cases} x & \text{if } x \text{ is odd,} \\ x/2 & \text{if } x \text{ is even.} \end{cases}$$

Let a be a positive integer.

Suppose a is odd. Then $g(a) = a$.

Suppose a is even. Then $2a \in \mathbb{Z}_{>0}$ and $g(2a) = a$.

Thus, g is onto.

On the other hand, $g(6) = g(3) = 3$, so g is not one-to-one. ■

5. *Use induction to prove that $n! > n^2$ for all integers $n \geq 4$.*

Proof: Define $P(n)$ to be the statement "$n! > n^2$".

Base case: Let $n = 4$. Then $n! = 24 > 16 = n^2$, so $P(4)$ is true.

Induction step: Suppose $P(n)$ is true for some $n = x \geq 4$.

Then $x! > x^2$, i.e. $\dfrac{x!}{x^2} > 1$.

Then

$$\begin{aligned} \frac{(x+1)!}{(x+1)^2} &= \frac{(x+1)x!}{(x+1)^2 \frac{x^2}{x^2}} \\ &= \frac{(x+1)x^2}{(x+1)^2} \left(\frac{x!}{x^2}\right) \\ &= \frac{x^2}{x+1} \left(\frac{x!}{x^2}\right) \\ &= \left(x - 1 + \frac{1}{x+1}\right) \left(\frac{x!}{x^2}\right) \end{aligned}$$

We note that $x - 1 + \frac{1}{x+1} \geq 4 - 1 + 0 = 3 > 1$, and so, since $\frac{x!}{x^2} > 1$,

$$\frac{(x+1)!}{(x+1)^2} > 1$$

i.e., $(x+1)! > x^2$. Thus, $P(x+1)$ is true.

Thus, $P(x)$ implies $P(x+1)$, and since $P(4)$ is true, by induction $P(n)$ is true for all integers $n \geq 4$. ■

6. *Let R be the relation defined on the real numbers, $\mathbb{R}$, by*

$$(x, y) \in R \Leftrightarrow \textit{there exist positive integers } n \textit{ and } m \textit{ such that } x^n = y^m.$$

Prove that R is an equivalence relation.

Proof: Let $x \in \mathbb{R}$. Then $x^1 = x^1$, and so $(x, x) \in R$.

Hence, R is reflexive.

Suppose $(x, y) \in R$. Then $x^m = y^n$ for some integers m and n.

Then, $y^n = x^m$, and so $(y, x) \in R$.

Hence R is symmetric.

Suppose $(x, y) \in R$ and $(y, z) \in R$.

Then $x^m = y^n$ and $y^r = z^s$ for positive integers m, n, r, and s.

Then $x^{rm} = y^{rn}$ and $y^{rn} = z^{sn}$, so $x^{rm} = z^{sn}$.

Since rm and sn are positive integers, we conclude that $(x, z) \in R$.

Hence, R is transitive.

Since R is reflexive, symmetric and transitive, R is an equivalence relation. ■