Math 300 A - Spring 2012
Final Exam
June 4, 2012
Solutions

1. Assign "true" or "false" to each of the following statements. No justification need be given.

   (a) F

   (b) T

   (c) T

   (d) T

   (e) F

   (f) F

   (g) T

   (h) T

2. Let $A = \mathcal{P}(\mathbb{R})$. Define $f : \mathbb{R} \to A$ by the formula

$$f(x) = \{y \in \mathbb{R} : y^2 < x\}.$$

   (a) Is f one-to-one? Prove your answer.
   Since $f(0) = \emptyset$, and $f(-1) = \emptyset$, and $0 \neq -1$, f is not one-to-one.

   (b) Is f onto? Prove your answer.
   No.
   Suppose $a > 0$. Then $f(a) = (-\sqrt{a}, \sqrt{a})$. So, $0 \in f(a)$.
   Now, suppose $a < 0$. Then $f(a) = \emptyset$.
   Hence, $f(a)$ is either the empty set, or $f(a)$ contains zero.
   Consider $T = \{1, 2\} \in A$. We know that $T \neq \emptyset$ and $0 \notin T$. Hence, $T \neq f(a)$ for any $a \in \mathbb{R}$.
   Thus, f is not onto.

3. Let R be a relation on $\mathbb{Q}$ defined by $(p/q, r/s) \in R \Leftrightarrow ps = qr$. Show that R is an equivalence relation.

   Reflexive:

   For any $p/q \in \mathbb{Q}$, $pq = qp$, $(p/q, p/q) \in R$. Hence, R is reflexive.

   Symmetric:

   Suppose $(a/b, c/d) \in R$. Then $ad = bc$. Hence, $bc = ad$ and $cb = da$, so $(c/d, a/b) \in R$. Thus, R is symmetric.

   Transitive:

   Suppose $(a_1/b_1, a_2/b_2) \in R$ and $(a_2/b_2, a_3/b_3) \in R$.

   Then $a_1b_2 = a_2b_1$ and $a_2b_3 = a_3b_2$.

   Then $a_1b_2b_3 = a_2b_1b_3$

   so

$a_1 b_3 b_2 = b_1 a_2 b_3$

so

$a_1 b_3 b_2 = b_1 a_3 b_2$

and hence

$a_1 b_3 = b_1 a_3$. Hence, $(a_1/b_1, a_3/b_3) \in R$. Thus, R is transitive.

Since R is reflexive, symmetric and transitive, R is an equivalence relation.

4. Give a proof by induction that 6 divides $n^3 - n$ for all $n \in \mathbb{Z}_{\geq 0}$.

Let $n = 0$. Then $n^3 - n = 0$ and $6|0$, so $6|n^3 - n$ when $n = 0$.

Suppose $6|k^3 - k$ for some $k \geq 0$.

Then

$$\begin{aligned} (k+1)^3 - (k+1) = \\ k^3 + 3k^2 + 3k + 1 - k - 1 = \\ k^3 + 3k^2 + 2k = \\ k^3 - k + 3k^2 + 3k = \\ k^3 - k + 3(k^2 + k) = \\ k^3 - k + 3k(k+1). \end{aligned}$$

For all $k \in \mathbb{Z}$, k is even or $k + 1$ is even. Hence, $k(k + 1)$ is even, and so $6|3k(k + 1)$.

Since $6|k^3 - k$, $6|(k + 1)^3 - (k + 1)$.

Hence, $6|n^3 - n$ for all $n \geq 0$.

5. Suppose $f : A \to C$ and $g : B \to C$. Prove that if A and B are disjoint, then

$$f \cup g : A \cup B \to C.$$

Let $x \in A \cup B$. Since $A \cap B = \emptyset$, $x \in A$ or $x \in B$ and not both.

If $x \in A$, then $\exists c \in C$ such that $(x, c) \in f$.

If $x \in B$, then $\exists c \in C$ such that $(x, c) \in g$.

Thus, $\exists c \in C$ such that $(x, c) \in f \cup g$.

Suppose $\exists c_1, c_2 \in C$ and $x \in A \cup B$ such that

$$(x, c_1) \in f \cup g \text{ and } (x, c_2) \in f \cup g.$$

Then three cases are possible:

$$\begin{aligned} &(1)\ (x, c_1) \in f \text{ and } (x, c_2) \in f \\ &(2)\ (x, c_1) \in f \text{ and } (x, c_2) \in g \\ &(3)\ (x, c_1) \in g \text{ and } (x, c_2) \in g \end{aligned}$$

Case (1) would imply that f is not a function.

Case (3) would imply that g is not a function.

Case (2) would imply that $x \in A$ and $x \in B$, which is not possible since $A \cap B = \emptyset$.

Thus, for all $x \in A \cup B$, there exists a unique $c \in C$ such that $(x, c) \in f \cup g$.

That is, $f \cup g : A \cup B \to C$.

6. Suppose R and S are equivalence relations on a set A and $A/R = A/S$. Prove that $R = S$.

   Let's show $R \subseteq S$.

   Suppose $(a, b) \in R$.

   Then $\exists x \in A/R$ such that $a \in x$ and $b \in x$.

   Hence, $x \in A/S$ and $a \in x$ and $b \in x$.

   Thus, $(a, b) \in S$ and so $R \subseteq S$.

   A symmetric argument shows that $S \subseteq R$, and so $R = S$.