Math 300 B - Spring 2012
Final Exam
June 6, 2012
Solutions

1. Assign ”true” or ”false” to each of the following statements. No justification need be given.

   (a) T

   (b) F

   (c) F

   (d) T

   (e) T

   (f) F

   (g) T

   (h) F

2. Suppose $f : A \to B$ and f is one-to-one. Prove that there is some set $B' \subseteq B$ such that $f^{-1} : B' \to A$.

   Let $B' = \{b \in B : \exists a \in A \text{such that} f(a) = b\}$. Consider $g : A \to B'$ defined by $g(a) = f(a)$ for $a \in A$.

   Let $b \in B'$. Then $\exists a \in A$ with $g(a) = b$. So g is onto.

   Suppose $a_1, a_2 \in A$ and $g(a_1) = g(a_2)$. Then $f(a_1) = f(a_2)$, so $a_1 = a_2$ since f is one-to-one. Hence g is one-to-one.

   Thus g is a bijection, so $g^{-1} : B' \to A$.

   But, $g^{-1} = f^{-1}$, so $f^{-1} : B' \to A$.

3. Let $A = \mathcal{P}(\mathbb{R})$. Define $f : \mathbb{R} \to A$ by the formula

$$f(x) = \{y \in \mathbb{R} : y^2 < x\}.$$

   (a) Is f one-to-one? Prove your answer.
   Since $f(0) = \emptyset$, and $f(-1) = \emptyset$, and $0 \neq -1$, f is not one-to-one.

   (b) Is f onto? Prove your answer.
   No.
   Suppose $a > 0$. Then $f(a) = (-\sqrt{a}, \sqrt{a})$. So, $0 \in f(a)$.
   Now, suppose $a < 0$. Then $f(a) = \emptyset$.
   Hence, $f(a)$ is either the empty set, or $f(a)$ contains zero.
   Consider $T = (1, 2) \in A$. We know that $T \notin \emptyset$ and $0 \notin T$. Hence, $T \neq f(a)$ for any $a \in \mathbb{R}$.
   Thus, f is not onto.

4. Let $S = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x - y \in \mathbb{Z}\}$.

   Is S an equivalence relation? Prove your answer.

   Yes.

Reflexive:

For all $x \in \mathbb{R}$, $x - x = 0 \in \mathbb{Z}$. So, $(x,x) \in R$ for all $x \in \mathbb{R}$. Hence, R is reflexive.

Symmetric:

Suppose $x, y \in \mathbb{R}$ and $x - y = m \in \mathbb{Z}$.

Then $y - x = -m \in \mathbb{Z}$.

So, $(x,y) \in R$ implies $(y,x) \in R$. Thus, R is symmetric.

Transitive:

Suppose $(x,y) \in R$ and $(y,z) \in R$.

Let $x - y = a$ and $y - z = b$. Then $a \in \mathbb{Z}$ and $b \in \mathbb{Z}$.

Then $x - z = a + b \in \mathbb{Z}$ since $\mathbb{Z}$ is closed under addition.

Hence $(x,z) \in R$, and so R is transitive.

Thus R is an equivalence relation.

5. Use induction to prove that 49 divides $36^n + 14n - 1$ for all $n \in \mathbb{Z}_{\geq 0}$.

If $n = 0$, then $36^n + 14n - 1 = 1 + 0 - 1 = 0$, so $49 | 36^n + 14n - 1$.

Suppose $49 | 36^k + 14k - 1$ for some $k \in \mathbb{Z}_{\geq 0}$. Say $36^k + 14k - 1 = 49m$ for some $m \in \mathbb{Z}$.

Then

$$\begin{aligned}
36^{k+1} + 14(k+1) - 1 &= \\
36(36^k + 14k - 1) - 36(14k) + 36 + 14(k+1) - 1 &= \\
36(49m) - 36(14k) + 14k + 36 + 14 - 1 &= \\
36(49m) - 35(14k) + 49 &= \\
36(49m) - 49(10k) + 49 &= \\
49(36m - 10k + 1).
\end{aligned}$$

Hence, $49 | 36^{k+1} + 14(k+1) - 1$.

Thus, $49 | 36^n + 14(n) - 1$ for all $n \in \mathbb{Z}_{\geq 0}$.

6. Suppose R is an equivalence relation on a set A.

Prove that for every $x \in A$ and $y \in A$, $y \in [x]_R$ iff $[y]_R = [x]_R$.

$(\leftarrow)$

Suppose $[y]_R = [x]_R$. Since $y \in [y]_R$, $y \in [x]_R$.

$(\rightarrow)$

Suppose $y \in [x]_R$. Then $(x,y) \in R$. Let $a \in [y]_R$. Then $(a,y) \in R$.

But $(x,y) \in R$, so $(y,x) \in R$, so by transitivity of R, $(a,x) \in R$. Hence, $a \in [x]$. Thus, $[y]_R \subseteq [x]_R$.

Let $b \in [x]_R$. Then $(x,b) \in R$. But, $(y,x) \in R$. By transitivity, $(y,b) \in R$, so $b \in [y]_R$.

So, $[x]_R \subseteq [y]_R$.

Thus, $[x]_R = [y]_R$.