Math 300 B - Spring 2013
Final Exam
June 12, 2013
Answers

1. *Let S be a set.*

   Define a function $f : \mathcal{P}(S) \to \mathcal{P}(S)$ by $f(A) = S \setminus A$ for all $A \in \mathcal{P}(S)$.

   Prove that f is a bijection.

   Proof: Let S be a set and define $f : \mathcal{P}((S) \to \mathcal{P}(S)$ as above.

   Suppose $A_1, A_2 \in \mathcal{P}(S)$ with $f(A_1) = f(A_2)$. Then

   $$S \setminus A_1 = S \setminus A_2.$$

   Suppose $x \in A_1$ but $x \notin A_2$.

   Then $x \in S$ (since $A_1 \subseteq S$), and $x \in S \setminus A_2$, but $x \notin S \setminus A_1$.

   This is a contradiction to the fact that $S \setminus A_1 = S \setminus A_2$.

   Hence if $x \in A_1$ then $x \in A_2$; i.e, $A_1 \subseteq A_2$.

   By an identical argument, we may also conclude that $A_2 \subseteq A_1$.

   Thus $A_1 = A_2$.

   Hence, $f(A_1) = f(A_2)$ implies $A_1 = A_2$, and so f is one-to-one.

   Suppose $B \in \mathcal{P}(S)$.

   Let $Z = S \setminus B$ (note that, since $Z \subseteq S, Z \in \mathcal{P}(S)$).

   Then $f(Z) = S \setminus (S \setminus B)$. We want to show that $f(Z) = B$.

   Suppose $x \in S \setminus (S \setminus B)$.

   Then $x \in S$ and $x \notin S \setminus B$.

   Since $x \notin S \setminus B$, and $x \in S$, $x \in B$.

   So $S \setminus (S \setminus B) \subseteq B$.

   On the other hand, suppose $x \in B$.

   Then $x \in S$ and $x \notin S \setminus B$, so $x \in S \setminus (S \setminus B)$.

   Hence, $S \setminus (S \setminus B) = B$.

   Thus, for all $B \in \mathcal{P}(S)$, $\exists Z \in \mathcal{P}(S)$ such that $f(Z) = B$.

   Thus f is onto.

   Hence f is one-to-one and onto, i.e., f is a bijection. ■

2. *Let S be the set of all functions $f : \mathbb{R} \Rightarrow \mathbb{R}$. Define a relation R on S by*

$$(f, g) \in R \Leftrightarrow \exists c \in \mathbb{R}, c \neq 0, \text{ such that } f(x) = cg(x) \text{ for all } x \in \mathbb{R}.$$

(a) *Prove that R is an equivalence relation.*

(b) *Let $f : \mathbb{R} \to \mathbb{R}$ be a non-constant, linear function.*
Consider $[f]$, the equivalence class of f under the relation R.
Let $g : \mathbb{R} \to \mathbb{R}$ be a linear function.
Show that, $g \in [f]$ iff f and g have the same x-intercept.

(a) **Proof:** Let $f \in S$.

Since $f(x) = (1)f(x)$ for all $x \in \mathbb{R}$, $(f, f) \in R$.

Hence, R is reflexive.

Suppose $(f, g) \in R$.

Then there exists a $c \in \mathbb{R}, c \neq 0$ such that $f(x) = cg(x)$ for all $x \in \mathbb{R}$. Since $c \neq 0$,

$$g(x) = \frac{1}{c}f(x)$$

for all $x \in \mathbb{R}$ and $\frac{1}{c} \neq 0, \frac{1}{c} \in \mathbb{R}$.

Thus $(g, f) \in R$.

Hence, R is symmetric.

Suppose $(f, g) \in R$ and $(g, h) \in R$.

Then there exist non-zero $c, d \in \mathbb{R}$ such that $f(x) = cg(x)$ and $g(x) = dh(x)$ for all $x \in \mathbb{R}$.

Hence, $f(x) = cdh(x)$ for all $x \in \mathbb{R}$.

Since c and d are non-zero, cd is non-zero, and $cd \in \mathbb{R}$, so $(f, h) \in R$.

Thus, R is transitive, and so R is an equivalence relation. ■

(b) **Proof:** Suppose $g \in [f]$.

Then $(g, f) \in R$ so there exists a non-zero $c \in \mathbb{R}$ such that $g(x) = cf(x)$ for all $x \in \mathbb{R}$.

Since g is non-constant, it has an x-intercepts; suppose $g(z) = 0$.

Then $0 = g(z) = cf(z)$, and since $c \neq 0$, $f(z) = 0$.

Hence, g and f have the same x-intercept.

Now, suppose g and f have the same x-intercept; suppose $g(a) = f(a) = 0$.

Suppose $g(x) = m_1x + b_1$ and $f(x) = m_2x + b_2$.

Then $m_1a + b_1 = 0$ and $m_2a + b_2 = 0$, so

$$a = -\frac{b_1}{m_1} = -\frac{b_2}{m_2},$$

so $b_1 = \frac{m_1b_2}{m_2}$.

Let $c = \frac{m_1}{m_2}$.

Then $cf(x) = \frac{m_1}{m_2}(m_2x + b_2) = m_1x + b_2\frac{m_1}{m_2} = m_1x + b1 = g(x)$ for all $x \in \mathbb{R}$.

Hence $g \in [f]$. ■

3. *Let A, B, and C be sets.*

   Prove that $A \cup C \subseteq B \cup C$ iff $A \setminus C \subseteq B \setminus C$.

   Proof: Suppose $A \cup C \subseteq B \cup C$.

   Suppose $x \in A \setminus C$.

   Then $x \in A$, so $x \in A \cup C$, and since $A \cup C \subseteq B \cup C$, $x \in B \cup C$.

   Since $x \in A \setminus C$, $x \notin C$.

   Since $x \in B \cup C$, but $x \notin C$, $x \in B$ and so $x \in B \setminus C$.

   Hence $A \setminus C \subseteq B \setminus C$.

   Now, suppose $A \setminus C \subseteq B \setminus C$.

   Suppose $x \in A \cup C$.

   If $x \in C$ then $x \in B \cup C$.

   Suppose $x \notin C$.

   Then $x \in A$ and so $x \in A \setminus C$.

   Since $A \setminus C \subseteq B \setminus C$, $x \in B \setminus C$, and so $x \in B$, and hence $x \in B \cup C$.

   Thus, $A \cup C \subseteq B \cup C$. ■

4. *Let A and B be sets.*

   Let f and g be functions from A to B.

   Prove that if $f \cap g \neq \varnothing$, then $f \setminus g$ is not a function from A to B.

   Proof: Suppose $f \cap g \neq \varnothing$.

   Then $\exists$ $(a, b) \in f \cap g$.

   So $(a, b) \notin f \setminus g$.

   Suppose $\exists(a, c) \in f \setminus g$ for some $c \in B$.

   Then $(a, c) \in f$ and, since f is a function $c = b$, so $(a, b) \in f \setminus g$.

   This is a contradiction: $(a, b) \notin f \setminus g$.

   Hence, there does not exist an ordered pair $(a, c) \in f \setminus g$ for any $c \in B$.

   Therefore, $f \setminus g$ is not a function. ■

5. *Let $n \in \mathbb{Z}_{>0}$.*

   Use induction to prove $\displaystyle\sum_{i=1}^{n} \frac{1}{(2i-1)(2i+1)} = \frac{n}{2n+1}$.

   Proof: Let $P(n)$ be the statement "$\displaystyle\sum_{i=1}^{n} \frac{1}{(2i-1)(2i+1)} = \frac{n}{2n+1}$".

   Since

   $$\sum_{i=1}^{1} \frac{1}{(2i-1)(2i+1)} = \frac{1}{3} = \frac{1}{2(1)+1},$$

   P(1) is true.

Suppose there exists a $k > 0$ such that $P(k)$ is true.

Then

$$\begin{aligned}
\sum_{i=1}^{k+1} \frac{1}{(2i-1)(2i+1)} &= \sum_{i=1}^{k} \frac{1}{(2i-1)(2i+1)} + \frac{1}{(2(k+1)-1)(2(k+1)+1)} \\
&= \frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)} \text{ (by the induction hypothesis)} \\
&= \frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)} \\
&= \frac{k(2k+3)+1}{(2k+1)(2k+3)} \\
&= \frac{2k^2+3k+1}{(2k+1)(2k+3)} \\
&= \frac{(2k+1)(k+1)}{(2k+1)(2k+3)} \\
&= \frac{k+1}{2k+3}
\end{aligned}$$

Hence $P(k+1)$ is true, so $P(k)$ implies $P(k+1)$.

Hence, by induction, $P(k)$ is true for all $k > 0$. ■

6. *Let n be an integer. Prove that $4|n^4 + 2n$ iff n is even.*

Proof:

Suppose n is even.

Then $n = 2m$ for some integer m, and

$$n^4 + 2n = (2m)^4 + 2(2m) = 16m^4 + 4m = 4(4m^4 + m)$$

and so $4|n^4 + 2n$.

Now suppose n is odd.

Then $n \equiv 1 \pmod 4$ or $n \equiv 3 \pmod 4$

If $n \equiv 1 \pmod 4$, then $n^4 \equiv 1 \pmod 4$ and $2n \equiv 2 \pmod 4$ and so $n^4 + 2n \equiv 3 \pmod 4$, in which case, 4 does not divide $n^4 + 2n$.

If $n \equiv 3 \pmod 4$, then $n^4 \equiv 1 \pmod 4$ and $2n \equiv 2 \pmod 4$, and so $n^4 + 2n \equiv 3 \pmod 4$, in which case 4 does not divide $n^4 + 2n$.

Thus, 4 divides $n^4 + 2n$ iff n is even.

■