Math 300 C - Winter 2013
Final Exam
June 13, 2013
Answers

1. *Suppose A, B, and C are sets with $A \cap B \subseteq C$. Prove that if $a \in B$, then $a \notin A \setminus C$.*

   Proof:

   Suppose $a \in B$.

   Suppose $a \in A \setminus C$.

   Then $a \in A$ and $a \notin C$.

   Since $a \in A$, $a \in A \cap B$.

   Since $A \cap B \subseteq C$, $a \in C$.

   This is contradicts our earlier conclusion that $a \notin C$.

   Hence, we conclude that $a \notin A \setminus C$. ■

2. *Let S be the set of all functions from $\mathbb{R}$ to $\mathbb{R}$.*

   Define a relation R on S by

   $$(f, g) \in R \Leftrightarrow \exists k \in \mathbb{R} \text{ such that } \; f(x) = g(x) \, \forall x \geq k.$$

   Prove that R is an equivalence relation.

   Proof: Suppose $f \in S$.

   Let $k = 0$ (this is arbitrary: any real number works here).

   Since $f(x) = f(x)$ for all $x \geq k$, $(f, f) \in R$.

   So R is reflexive.

   Suppose $(f, g) \in R$.

   Then there exists a $k \in \mathbb{R}$ such that $f(x) = g(x)$ for all $x \geq k$.

   So $g(x) = f(x)$ for all $x \geq k$, and hence $(g, f) \in R$.

   So R is symmetric.

   Suppose $(f, g) \in R$ and $(g, h) \in R$.

   Then there exists a $k \in \mathbb{R}$ such that $f(x) = g(x)$ for all $x \geq k$, and there exists a $j \in \mathbb{R}$ such that $g(x) = h(x)$ for all $x \geq m$.

   Let m equal the maximum of k and j.

   Suppose $x \geq m$.

   Then $x \geq k$ and $x \geq j$.

   Hence $f(x) = g(x)$ and $g(x) = h(x)$, so $f(x) = h(x)$.

   Thus $f(x) = h(x)$ for all $x \geq m$, so $(f, h) \in R$.

   Thus R is transitive, and so R is an equivalence relation. ■

3. *Using induction, prove that*

$$(1) + (1+2) + (1+2+3) + \cdots + (1+2+3+\cdots+n) = \frac{1}{6}n(n+1)(n+2)$$

for all positive integers, n.

Proof: Let $P(n)$ be the statement

$$\text{“}(1) + (1+2) + (1+2+3) + \cdots + (1+2+3+\cdots+n) = \frac{1}{6}n(n+1)(n+2)\text{”}.$$

Since $1 = \frac{1}{6}(1)(1+1)(1+2)$, $P(1)$ is true.

Suppose that there exists a $k \geq 1$ such that $P(k)$ is true.

Then

$$(1) + (1+2) + (1+2+3) + \cdots + (1+2+3+\cdots+k) + (1+2+3+\cdots+k+(k+1))$$
$$= \frac{1}{6}k(k+1)(k+2) + (1+2+3+\cdots+k+(k+1)) \text{ (by our induction hypothesis)}$$

In class, we showed that $1+2+3+\cdots+k = \frac{1}{2}k(k+1)$, so we have

$$\frac{1}{6}k(k+1)(k+2) + (1+2+3+\cdots+k+(k+1)) =$$
$$\frac{1}{6}k(k+1)(k+2) + \frac{1}{2}(k+1)(k+2) =$$
$$\frac{1}{6}(k+1)(k+2)(k+3).$$

Hence $P(k+1)$ is true.

Thus $P(k)$ implies $P(k+1)$, and so, by induction, $P(n)$ is true for all $n \geq 0$. ■

4. *Suppose $\mathcal{M}, \mathcal{N}$, and $\mathcal{P}$ are families of sets, with $\mathcal{M} \neq \varnothing$, $\mathcal{N} \neq \varnothing$, and $\mathcal{P} \neq \varnothing$.*

   Suppose that for every $A \in \mathcal{M}$ and $B \in \mathcal{N}$, $A \cup B \in \mathcal{P}$.

   Prove that $\cap\mathcal{P} \subseteq (\cap\mathcal{M}) \cup (\cap\mathcal{N})$.

   Proof: Suppose $x \in \cap\mathcal{P}$.

   So $x \in C$ for all $C \in \mathcal{P}$.

   Suppose $x \notin \cap\mathcal{M}$.

   Then $x \notin A$ for some $A \in \mathcal{M}$.

   Suppose there exists a $B \in \mathcal{N}$ such that $x \notin B$.

   Now, $A \cup B \in P$, so $x \in A \cup B$, and, since $x \notin B$, $x \in A$.

   But $x \notin A$, so this is a contradiction.

   Hence, there does not exist a $B \in \mathcal{N}$ such that $x \notin B$.

   That is, $x \in B$ for all $B \in \mathcal{N}$.

   Hence, $x \in \cap\mathcal{N}$.

   Thus $x \notin \cap\mathcal{M}$ implies $x \in \cap\mathcal{N}$, and so $x \in (\cap\mathcal{M}) \cup (\cap\mathcal{N})$.

   Hence, $x \in \cap\mathcal{P}$ implies $x \in (\cap\mathcal{M}) \cup (\cap\mathcal{N})$.

   Therefore, $\cap\mathcal{P} \subseteq (\cap\mathcal{M}) \cup (\cap\mathcal{N})$. ■

5. *Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ be defined by*

$$f(a, b) = (5a + 4b, a - 2b).$$

Prove that f is a bijection.

Proof: Suppose $f(a_1, b_1) = f(a_2, b_2)$.

Then

$$5a_1 + 4b_1 = 5a_2 + 4b_2$$

and

$$a_1 - 2b_1 = a_2 - 2b_2.$$

From the second equation, we have

$$2a_1 - 4b_1 = 2a_2 - 4b_2,$$

which, when added to the first equation yields

$$7a_1 = 7a_2$$

from which we conclude that $a_1 = a_2$.

The first equation then yields

$$4b_1 = 4b_2$$

so $b_1 = b_2$.

Thus, $(a_1, b_1) = (a_2, b_2)$, so f is one-to-one.

Let $(a, b) \in \mathbb{R} \times \mathbb{R}$.

Let $x = \frac{1}{7}(a + 2b)$ and $y = \frac{1}{14}(a - 5b)$.

Then

$$\begin{aligned} f(x, y) = (5x + 4y, x - 2y) &= \left(\frac{5}{7}(a + 2b) + \frac{4}{14}(a - 5b), \frac{1}{7}(a + 2b) - \frac{2}{14}(a - 5b)\right) \\ &= (a, b). \end{aligned}$$

Thus, f is onto, so f is a bijection. ■

6. *Prove that, for all $n \in \mathbb{Z}$, $4|3n^2 + 2n + 3$ iff n is odd.*

Proof: Suppose n is odd.

Then $n = 2m + 1$ for some $m \in \mathbb{Z}$.

Then

$$\begin{aligned} 3n^2 + 2n + 3 = 3(2m + 1)^2 + 2(2m + 1) + 3 &= 3(4m^2 + 4m + 1) + 4m + 5 \\ &= 12m^2 + 12m + 3 + 4m + 5 \\ &\equiv 0 \pmod{4}. \end{aligned}$$

In other words, $4|3n^2 + 2n + 3$.

Now, suppose n is even. So $n = 2m$ for some $m \in \mathbb{Z}$.

Then

$$\begin{aligned} 3n^2 + 2n + 3 &= 3(2m)^2 + 2(2m) + 3 \\ &= 12m^2 + 4m + 3 \\ &\equiv 3 \pmod{4}. \end{aligned}$$

Since $3 \not\equiv 0 \pmod{4}$, 4 does not divide $3n^2 + 2n + 3$.

Thus, $4|3n^2 + 2n + 3$ iff n is odd. ■