Math 300 C - Spring 2015
Final Exam
June 11, 2015

Name: ______________________ Student ID no. : ______________

Signature: ____

1	10	
2	10	
3	10	
4	15	
5	10	
6	10	
Total	65	

- Complete all 6 questions.
- You have 110 minutes to complete the exam.

1. Let $f : \mathbb{R} \to \mathbb{R}_{\geq 0}$ be a bijection. Define $g : \mathbb{R} \to \mathbb{R}_{\geq 0}$ by

$$g(x) = f(x)^2$$

for all $x \in \mathbb{R}$.

Prove that g is a bijection.

2. Use induction to prove that, for all positive integers n,

$$\sum_{j=1}^{n}(j^2+1)j! = n(n+1)!$$

3. On $\mathbb{Z} \times \mathbb{Z}$, define a relation R by

$$((a, b), (c, d)) \in R \text{ iff } a + d = b + c.$$

Prove that R is an equivalence relation.

4. Let A, B and C be sets.

Suppose $f : A \to B$, $g : B \to C$ and $h : B \to C$.

(a) Prove that if f is onto and $g \circ f = h \circ f$, then $g(x) = h(x)$ for all $x \in B$.

(b) Suppose $g \circ f$ is one-to-one. Is f necessarily one-to-one? Is g necessarily one-to-one? Prove your answers.

5. Let $\mathcal{F}$ and $\mathcal{G}$ be families of sets.

   (a) Prove that

$$\cup\mathcal{F} \setminus \cup\mathcal{G} \subseteq \cup(\mathcal{F} \setminus \mathcal{G}).$$

   (b) Give an example of non-empty families $\mathcal{F}$ and $\mathcal{G}$ such that

$$\cup\mathcal{F} \setminus \cup\mathcal{G} \neq \cup(\mathcal{F} \setminus \mathcal{G}).$$

   Prove that your example is valid.

6. A positive integer n is called a *triangular number* if

$$n = \frac{1}{2}k(k+1)$$

for some positive integer k.

The smallest ones are $1, 3, 6, 10, 15$.

Show that there are infinitely many positive integers that are not are the sum of two triangular numbers (hint: consider the situation modulo 9).

Axioms of the Integers (AIs)

Suppose a, b, and c are integers.

- **Closure:**

  $a + b$ and ab are integers.

- **Substitution of Equals:**

  If $a = b$, then $a+c = b+c$ and $ac = bc$.

- **Commutativity:**

  $a + b = b + a$ and $ab = ba$.

- **Associativity:**

  $(a + b) + c = a + (b + c)$ and $(ab)c = a(bc)$.

- **The Distributive Law:**

  $a(b + c) = ab + ac$

- **Identities:**

  $a + 0 = 0 + a = a$ and $a \cdot 1 = 1 \cdot a = a$

  0 is called the *additive identity*

  1 is called the *multiplicative identity*.

- **Additive Inverses:**

  There exists an integer $-a$ such that $a + (-a) = (-a) + a = 0$.

- **Trichotomy:**

  Exactly one of the following is true: $a < 0$, $-a < 0$, or $a = 0$.

Sets

$A \subseteq B$ iff $x \in A$ implies $x \in B$

$A = B$ iff $A \subseteq B$ and $B \subseteq A$

$x \in A \cup B$ iff $x \in A$ or $x \in B$

$x \in A \cap B$ iff $x \in A$ and $x \in B$

$x \in A \setminus B$ iff $x \in A$ and $x \notin B$

$\mathcal{P}(A)$ is the set of all subsets of a set A

Elementary Properties of the Integers (EPIs)

Suppose a, b, c, and d are integers.

1. $a \cdot 0 = 0$
2. If $a + c = b + c$, then $a = b$.
3. $-a = (-1) \cdot a$
4. $(-a) \cdot b = -(a \cdot b)$
5. $(-a) \cdot (-b) = a \cdot b$
6. If $a \cdot b = 0$, then $a = 0$ or $b = 0$.
7. If $a \leq b$ and $b \leq a$, then $a = b$.
8. If $a < b$ and $b < c$, then $a < c$.
9. If $a < b$, then $a + c < b + c$.
10. If $a < b$ and $0 < c$, then $ac < bc$.
11. If $a < b$ and $c < 0$, then $bc < ac$.
12. If $a < b$ and $c < d$, then $a + c < b + d$.
13. If $0 \leq a < b$ and $0 \leq c < d$, then $ac < bd$.
14. If $a < b$, then $-b < -a$.
15. $0 \leq a^2$
16. If $ab = 1$, then either $a = b = 1$ or $a = b = -1$.

NOTE: Properties 8-14 hold if each $<$ is replaced with $\leq$.

One theorem for reference:

Theorem DAS (Divisors are Smaller): Let a and b be positive integers. Then $a|b$ implies $a \leq b$.