Math 300 C - Spring 2015
Final Exam
June 11, 2015
Answers

1. Let $f : \mathbb{R} \to \mathbb{R}_{\geq 0}$ *be a bijection. Define* $g : \mathbb{R} \to \mathbb{R}_{\geq 0}$ *by*

$$g(x) = f(x)^2$$

for all $x \in \mathbb{R}$*.*

Prove that g *is a bijection.*

Suppose $g(x_1) = g(x_2)$ for some $x_1, x_2 \in \mathbb{R}$.

Then $f(x_1)^2 = f(x_2)^2$.

Then $f(x_1) = f(x_2)$ since $f(x) \geq 0$ for all $x \in \mathbb{R}$.

Since f is one-to-one, $x_1 = x_2$.

Thus, $g(x_1) = g(x_2)$ implies $x_1 = x_2$, so g is one-to-one.

Let $y \in \mathbb{R}_{\geq 0}$.

Let $z = \sqrt{y}$. Then $z \in \mathbb{R}_{\geq 0}$.

Then, since f is onto, $\exists \hat{x} \in \mathbb{R}$ such that $f(\hat{x}) = z$.

Then $g(\hat{x}) = (f(\hat{x}))^2 = z^2 = (\sqrt{y})^2 = y$.

Hence, for all $y \in \mathbb{R}_{\geq 0}$, $\exists x \in \mathbb{R}$ such that $g(x) = y$.

Thus, g is onto.

Therefore, g is one-to-one and onto, and so g is a bijection. ■

2. *Use induction to prove that, for all positive integers* n*,*

$$\sum_{j=1}^{n} (j^2 + 1)j! = n(n+1)!$$

Let $P(n) =$ “ $\sum_{j=1}^{n} (j^2 + 1)j! = n(n+1)!$”.

Base case. Let $n = 1$.

Then

$$\sum_{j=1}^{n} (j^2 + 1)j! = \sum_{j=1}^{1} (j^2 + 1)j! = (1^2 + 1)(1!) = 2$$

while

$$n(n+1)! = 1(1+1)! = 2$$

so $P(1)$ is true.

Induction step. Suppose $P(n)$ is true for some $n = x \geq 1$.

Then $P(x)$ is true.

Then

$$\begin{aligned}\sum_{j=1}^{x+1}(j^2+1)j! &= \sum_{j=1}^{x}(j^2+1)j! + ((x+1)^2+1)((x+1)!) \\ &= x(x+1)! = (x^2+2x+2)(x+1)! \\ &= (x+1)!(x^2+3x+2) \\ &= (x+1)!(x+2)(x+1) \\ &= (x+1)(x+2)! = (x+1)((x+1)+1)!\end{aligned}$$

so $P(x+1)$ is true.

Hence, $P(x)$ implies $P(x+1)$.

Then, since $P(1)$ is true, by induction $P(n)$ is true for all $n \geq 1$. ■.

3. *On $\mathbb{Z} \times \mathbb{Z}$, define a relation R by*

$$((a,b),(c,d)) \in R \text{ iff } a+d = b+c.$$

Prove that R is an equivalence relation.

Reflexive? Let $(x,y) \in \mathbb{Z} \times \mathbb{Z}$.

Then $x+y = y+x$, so $((x,y),(x,y)) \in R$.

So R is reflexive.

Symmetric? Suppose $((x_1,y_1),(x_2,y_2)) \in R$.

Then $x_1+y_2 = y_1+x_2$ and so $x_2+y_1 = y_2+x_1$, and so $((x_2,y_2),(x_1,y_1)) \in R$.

So R is symmetric.

Transitive? Suppose $((x_1,y_1),(x_2,y_2)) \in R$ and $((x_2,y_2),(x_3,y_3)) \in R$.

Then $x_1+y_2 = y_1+x_2$ and $x_2+y_3 = y_2+x_3$.

Summing, we have $x_1+y_2+x_2+y_3 = y_1+x_2+y_2+x_3$.

Subtracting x_2+y_2 from both sides, we get

$$x_1+y_3 = y_1+x_2$$

and hence $((x_1,y_1),(x_3,y_3)) \in R$.

So R is transitive.

Thus, R is reflexive, symmetric and transitive, so R is an equivalence relation.■

4. *Let A, B and C be sets.*

Suppose $f : A \to B$, $g : B \to C$ and $h : B \to C$.

(a) *Prove that if f is onto and $g \circ f = h \circ f$, then $g(x) = h(x)$ for all $x \in B$.*
Suppose $x \in B$.
Then $\exists a \in A$ such that $f(a) = x$ since f is onto.
Then $g(f(a)) = h(f(a))$ since $g \circ f = h \circ f$.
Hence, $g(x) = h(x)$. Thus, for all x in B, $g(x) = h(x)$. ■

(b) *Suppose $g \circ f$ is one-to-one. Is f necessarily one-to-one? Is g necessarily one-to-one? Prove your answers.*
Suppose f is not one-to-one.
Then there exist $x_1, x_2 \in A$ such that $f(x_1) = f(x_2)$ and $x_1 \neq x_2$.
Then $g(f(x_1)) = g(f(x_2))$ by $x_1 \neq x_2$.
Hence, $g \circ f$ is not one-to-one.
This contradicts the assumption that $g \circ f$ is one-to-one.
Hence, the assumption that f is not one-to-one is false.
Thus f is one-to-one.

The function g need not be one-to-one.
Let $A = \{a\}$, $B = \{b_1, b_2\}$ and $C = \{c\}$.
Define $f = \{(a, b_1)\}$, $g = \{(b_1, c), (b_2, c)\}$.
Then $g \circ f = \{(a, c)\}$ so $g \circ f$ is one-to-one, but g is not.■

5. *Let $\mathcal{F}$ and $\mathcal{G}$ be families of sets.*

(a) *Prove that*

$$\cup\mathcal{F} \setminus \cup\mathcal{G} \subseteq \cup(\mathcal{F} \setminus \mathcal{G}).$$

Suppose $x \in \cup\mathcal{F} \setminus \cup\mathcal{G}$.
So $x \in \cup\mathcal{F}$ and $x \notin \cup\mathcal{G}$.
Then $\exists A \in \mathcal{F}$ such that $x \in A$.
Suppose $A \in \mathcal{G}$.
 Then $x \in \cup\mathcal{G}$.
 This contradicts $x \notin \cup\mathcal{G}$
Hence, $A \notin \mathcal{G}$.
Thus, $A \in \mathcal{F} \setminus \mathcal{G}$, and so $x \in \cup(\mathcal{F} \setminus \mathcal{G})$.
Therefore, $x \in \cup\mathcal{F} \setminus \cup\mathcal{G}$ implies $x \in \cup(\mathcal{F} \setminus \mathcal{G})$, and so $\cup\mathcal{F} \setminus \cup\mathcal{G} \subseteq (\mathcal{F} \setminus \mathcal{G})$. ■

(b) *Give an example of non-empty families $\mathcal{F}$ and $\mathcal{G}$ such that*

$$\cup\mathcal{F} \setminus \cup\mathcal{G} \neq \cup(\mathcal{F} \setminus \mathcal{G}).$$

Prove that your example is valid.
Let $\mathcal{F} = \{\{1,2\}, \{2,3\}\}$ and $\mathcal{G} = \{\{2\}\}$.
Then $\mathcal{F} \setminus \mathcal{G} = \mathcal{F}$ and $\cup(\mathcal{F} \setminus \mathcal{G}) = \{1,2,3\}$, while

$$\cup\mathcal{F} = \{1,2,3\}, \text{ and } \cup\mathcal{G} = \{2\},$$

so $\cup\mathcal{F} \setminus \cup\mathcal{G} = \{1,3\} \neq \cup(\mathcal{F} \setminus \mathcal{G})$. ■

6. *A positive integer n is called a triangular number if*

$$n = \frac{1}{2}k(k+1)$$

for some positive integer k.

The smallest ones are $1, 3, 6, 10, 15$.

Show that there are infinitely many positive integers that are not are the sum of two triangular numbers (hint: consider the situation modulo 9).

First, we note that if $k = r + 9m$ for some integers k, r and m, then

$$\frac{1}{2}k(k+1) = \frac{1}{2}r(r+1) + 9mr + 9\left(\frac{m(9m+1)}{2}\right).$$

If m is even, then $\frac{m(9m+1)}{2}$ is an integer, since $2 \mid m$, while if m is odd, then $9m+1$ is even, and so $\frac{m(9m+1)}{2}$ is an integer.

Hence, $\frac{1}{2}k(k+1) \equiv \frac{1}{2}r(r+1) \pmod{9}$.

Thus, we make the following table:

k	$\frac{1}{2}k(k+1)$ reduced mod 9
0	0
1	1
2	3
3	6
4	1
5	6
6	3
7	1
8	0

and conclude that all triangular numbers are congruent to $0, 1, 3$ or 6 mod 9.

Hence, summing two triangular numbers involves only the following cases, summing pair-wise mod 9:

	0	1	3	6
0	0	1	3	6
1	1	2	4	7
3	3	4	6	0
6	6	7	0	3

So we see that the sum of two triangular numbers must be congruent to $0, 1, 2, 3, 4, 6$ or 7 mod 9.

This leaves out 5 (and 8).

So, any positive integer congruent to 5 mod 9 is not the sum of two triangular numbers.

Since all numbers of the form $5 + 9k$ are congruent to 5 mod 9, and we can choose k to be any of the infinite set of positive integers, we conclude that there are infinitely many positive integers that are not the sum of two triangular numbers. ■