Math 300 C - Spring 2016
Final Exam
June 9, 2016
Answers

1. *Prove the following two theorems.*

   (a) *Let $\mathcal{F}$ be a family of sets. Suppose $\varnothing \in \mathcal{F}$. Then $\bigcap \mathcal{F} = \varnothing$.*

   Proof:

   Suppose $\bigcap \mathcal{F} \neq \varnothing$.

   Suppose $x \in \bigcap \mathcal{F}$.

   Then $x \in S \forall S \in \mathcal{F}$.

   Suppose $\varnothing \in \mathcal{F}$.

   Then $x \in \varnothing$.

   This is a contradiction, since $\varnothing$ has no elements.

   Hence, the assumption $\varnothing \in \mathcal{F}$ is false, and so $\varnothing \notin \mathcal{F}$. ■

   (b) *Let A and B be sets. Suppose $A \subseteq B$. Then $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.*

   Proof:

   Suppose $S \in \mathcal{P}(A)$.

   Then $S \subseteq A$.

   Suppose $x \in S$.

   Then $x \in A$, so $x \in B$ since $A \subseteq B$.

   Thus, for all $x \in S$, $x \in B$, and so $S \subset B$.

   Thus, $S \in \mathcal{P}(B)$.

   Hence, $\forall T \in \mathcal{P}(A)$, $T \in \mathcal{P}(B)$, and so $\mathcal{P}(A) \subseteq \mathcal{P}(B)$. ■

2. *Use induction to prove that, for all integers $n > 0$,*

$$\sum_{i=1}^{n} i2^i = (n-1)2^{n+1} + 2.$$

Proof:

For all positive integers n, define $P(n)$ be the statement "$\sum_{i=1}^{n} i2^i = (n-1)2^{n+1} + 2.$ ".

We proceed by induction.

Base Case: Let $n = 1$. Then

$$\sum_{i=1}^{1} i2^i = 1 \cdot 2^1 = 2$$

while $(n-1)2^{n+1} + 2 = (0)(2^2) + 2 = 2$ and so $P(1)$ is true.

Induction Step: Suppose $P(n)$ is true for some $n = k \geq 0$.

So $P(k)$ is true, so

$$\sum_{i=1}^{k} i2^i = (k-1)2^{k+1} + 2.$$

Then

$$\begin{aligned}
\sum_{i=1}^{k+1} i2^i &= \sum_{i=1}^{k} i2^i + (k+1)2^{k+1} \\
&= (k-1)2^{k+1} + 2 + (k+1)2^{k+1} \\
&= 2k2^{k+1} + 2 \\
&= k2^{k+2} + 2 \\
&= (k+1-1)2^{(k+1)+1} + 2.
\end{aligned}$$

Thus, $P(k+1)$ is true.

Hence, $P(k)$ implies $P(k+1)$, and since $P(1)$ is true, by induction $P(n)$ is true for all $n > 0$. ■

3. *Let A and B be finite disjoint sets. Prove that* $|A \cup B| = |A| + |B|$.

Proof: Let A and B be disjoint sets. Let $n = |A|$ and $m = |B|$. Then there a bijection f from A to I_n and there is a bijection g from B to I_m.

Define $h : A \cup B \to I_{m+n}$ by

$$h(x) = \begin{cases} f(x) & \text{if } x \in A \\ g(x) + n & \text{if } x \in B. \end{cases}$$

Let $x \in A \cup B$.

Then, since A and B are disjoint, x is in exactly one of A and B.

Hence $h(x)$ is well-defined.

Further, for any $x \in A$, $1 \le f(x) \le n$, and for any $x \in B$, $n + 1 \le g(x) + n \le n + m$.

Hence, $h(x) \in I_{m+n}$, so indeed, $h : A \cup B \to I_{m+n}$.

Suppose $h(x_1) = h(x_2)$ for some x_1 and $x_2 \in A \cup B$.

Suppose $x_1 \in A$ and $x_2 \in B$. Then $h(x_1) \le n$ and $h(x_2) \ge n + 1$, so $h(x_1) \ne h(x_2)$. This is a contradiction, so it is not possible that $x_1 \in A$ and $x_2 \in B$.

Suppose $x_2 \in A$ and $x_1 \in B$. Then $h(x_2) \le n$ and $h(x_1) \ge n + 1$, so $h(x_2) \ne h(x_1)$. This is a contradiction, so it is not possible that Suppose $x_2 \in A$ and $x_1 \in B$.

Suppose $x_1, x_2 \in A$. Then $f(x_1) = f(x_2)$, so $x_1 = x_2$.

Suppose $x_1, x_2 \in B$. Then $g(x_1) + n = g(x_2) + n$, so $g(x_1) = g(x_2)$, and hence $x_1 = x_2$.

Thus, $x_1 = x_2$, so h is one-to-one.

Suppose $y \in I_{m+n}$.

Suppose $1 \le y \le n$. Then $y \in I_n$, and since f is bijective, there is a $\hat{x} \in A$ such that $f(\hat{x}) = y$ and $h(\hat{x}) = y$.

Suppose $y > n$. Then $y - n \in I_m$, and since g is bijective, there is a $\hat{x} \in B$ such that $g(\hat{x}) = y - n$ and $h(\hat{x}) = y$.

Thus, there is a $\hat{x} \in A \cup B$ such that $h(\hat{x}) = y$ and so h is surjective.

Therefore, h is a bijection.

Hence, $A \cup B \sim I_{m+n}$, and so $|A \cup B| = m + n = |A| + |B|$. ■

4. *Define a relation R on the set of integers $\mathbb{Z}$ by*

$$(a, b) \in R \text{ iff } |a^2 - b^2| \leq 6.$$

Determine whether or not R is (1) reflexive, (2) symmetric, and (3) transitive.

Prove your answers are correct.

(Note: you need to answer three separate questions here.)

Reflexivity: Let $a \in \mathbb{Z}$.

Then $|a^2 - a^2| = |0| = 0 \leq 6$, so $(a, a) \in R$.

Hence, R is reflexive.

Symmetry: Suppose $(a, b) \in R$.

Then $|a^2 - b^2| \leq 6$.

Hence $-6 \leq a^2 - b^2 \leq 6$, and so

$$6 \geq b^2 - a^2 \geq -6.$$

That is, $-6 \leq b^2 - a^2 \leq 6$, and so $|b^2 - a^2| \leq 6$.

Hence, $(b, a) \in R$, and so R is symmetric.

Transitive: Since $|3^2 - 2^2| = 5 \leq 6$ and $|2^2 - 1^2| = 3 \leq 6$, we conclude that $(3, 2) \in R$ and $(2, 1) \in R$.

On the other hand, $|3^2 - 1^2| = 8 \not\leq 6$, and so $(3, 1) \notin R$.

Hence, R is not transitive.

5. *Let A and B be sets, and $f : A \to B$ and $g : A \to B$.*

Prove that if $f \cap g \neq \varnothing$, then $f \setminus g$ is not a function from A to B.

Proof: Let A and B be sets, and $f : A \to B$ and $g : A \to B$.

Suppose $f \cap G \neq \varnothing$.

Then $\exists (a, b) \in f \cap g$.

So $(a, b) \in f$ and $(a, b) \in g$, and hence

$$(a, b) \notin f \setminus g. \quad (*)$$

Now, suppose $(a, x) \in f \setminus g$ for some $x \in B$.

Then $(a, x) \in f$.

Since f is a function, $f(a) = x = b$.

Hence, $(a, b) \in f \setminus g$.

This is a contradiction to $(*)$.

Hence, $(a, x) \notin f \setminus g$ for any $x \in B$.

Therefore, $f \setminus g$ is not a function from A to B. ■

6. *Prove that there are infinitely many positive integers that are not the sum of a cube and twice a square (i.e., a number of the form $2i^2$ where i is an integer).*

   Hint: consider the situation modulo 8.

   Proof:

   Let $n \in \mathbb{Z}$.

   Then, by the Euclidean Division theorem, $n \equiv 0, 1, 2, 3, 4, 5, 6,$ or $7 \pmod{8}$.

   So $n^3 \equiv 0, 1, 0, 3, 0, 5, 0,$ or $7 \pmod{8}$.

   That is, $n^3 \equiv 0, 1, 3, 5,$ or $7 \pmod{8}$.

   Let $m \in \mathbb{Z}$

   Then $m \equiv 0, 1, 2, 3, 4, 5, 6,$ or $7 \pmod{8}$.

   So, $m^2 \equiv 0, 1, 4, 1, 0, 1, 4,$ or $1 \pmod{8}$.

   Hence, $2m^2 \equiv 0$ or $2 \pmod{8}$.

   Applying these results, we find that we have the following possible values of $n^3 + 2m^2$ modulo 8:

	0	1	3	5	7
0	0	1	3	5	7
2	2	3	5	7	1

   We notice that 4 and 6 are missing from the table.

   Hence, if $x \equiv 4 \pmod{8}$, then x is not the sum of a cube and twice a square.

   Since every integer of the form $4 + 8k$ is congruent to 4 modulo 8, we conclude that there are infinitely many integers that are not the sum of a cube and twice and square. ■