Math 300 C - Spring 2022
Final Exam
June 9, 2022

Name: ______________________________ Student ID no. : ____________________

Signature: ____

- You have 110 minutes to complete the exam.
- When time is called, you must stop writing immediately.

1. Let A, B, and C be sets.

   Prove that $A \cup C \subseteq B \cup C$ iff $A \setminus C \subseteq B \setminus C$.

2. Let A be the set of all functions $f : \mathbb{R} \to \mathbb{R}$. Define a relation R on A by

$$(f, g) \in R \text{ iff there exists a } c \in \mathbb{R} \text{ such that } f(x) = g(x) + c \text{ for all } x \in \mathbb{R}.$$

Prove that R is an equivalence relation.

3. (a) Let $\mathcal{F}$ and $\mathcal{G}$ be families. Prove that $\bigcup \mathcal{F} \setminus \bigcup \mathcal{G} \subseteq \bigcup (\mathcal{F} \setminus \mathcal{G})$.

(b) Prove that there exist non-empty families $\mathcal{F}$ and $\mathcal{G}$ such that $\bigcup \mathcal{F} \setminus \bigcup \mathcal{G} \neq \bigcup (\mathcal{F} \setminus \mathcal{G})$.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^2 - 2x + 3 & \text{if } x \leq 1; \\ -3x + 5 & \text{if } x > 1. \end{cases}$$

Prove that f is a bijection.

5. Prove that $20! + 13$ cannot be written as $i^2 + j^4$ for any integers i and j. (Hint: consider the situation modulo 5).

6. Suppose A, B and C are sets. Suppose $f : A \to B$ and $g : B \to C$.

(a) Prove that if f is onto and g is not one-to-one, then $g \circ f$ is not one-to-one.

(b) Give an example to show that $g \circ f$ may be one-to-one when g is not one-to-one.

Elementary Properties of the Integers (EPIs)

Suppose a, b, c, and d are integers.

1. **Closure:** $a+b$ and ab are integers.
2. **Substitution of Equals:** If $a=b$, then $a+c=b+c$ and $ac=bc$.
3. **Commutativity:** $a+b=b+a$ and $ab=ba$.
4. **Associativity:** $(a+b)+c=a+(b+c)$ and $(ab)c=a(bc)$.
5. **The Distributive Law:** $a(b+c)=ab+ac$
6. **Identities:** $a+0=0+a=a$ and $a\cdot 1=1\cdot a=a$.

   0 is called the additive identity.

   1 is called the multiplicative identity.
7. **Additive Inverses:** There exists an integer $-a$ such that $a+(-a)=(-a)+a=0$.
8. **Trichotomy:** Exactly one of the following is true:
   $a>0$, $-a>0$, or $a=0$.
9. **The Well-Ordering Principle:** Every non-empty set of positive integers contains a smallest element.
10. $a\cdot 0=0$
11. If $a+c=b+c$, then $a=b$.
12. $-a=(-1)\cdot a$
13. $(-a)\cdot b=-(ab)$
14. $(-a)\cdot(-b)=ab$
15. If $ab=0$, then $a=0$ or $b=0$.
16. If $a\leq b$ and $b\leq a$, then $a=b$.
17. If $a<b$ and $b<c$, then $a<c$.
18. If $a<b$, then $a+c<b+c$.
19. If $a<b$ and $0<c$, then $ac<bc$.
20. If $a<b$ and $c<0$, then $bc<ac$.
21. If $a<b$ and $c<d$, then $a+c<b+d$.
22. If $0\leq a<b$ and $0\leq c<d$, then $ac<bd$.
23. If $a<b$, then $-b<-a$.
24. $0\leq a^2$, where $a^2=a\cdot a$.
25. If $ab=1$, then either $a=b=1$ or $a=b=-1$.

NOTE: Properties 17-23 hold if each $<$ is replaced with $\leq$.

One theorem for reference:
Theorem DAS (Divisors are Smaller): Let a and b be positive integers. Then $a|b$ implies $a\leq b$.

Sets

$$A\subseteq B \text{ iff } x\in A \text{ implies } x\in B$$

$$A=B \text{ iff } A\subseteq B \text{ and } B\subseteq A$$

$$x\in A\cup B \text{ iff } x\in A \text{ or } x\in B$$

$$x\in A\cap B \text{ iff } x\in A \text{ and } x\in B$$

$$x\in A\setminus B \text{ iff } x\in A \text{ and } x\notin B$$