Math 300 C - Spring 2022
Final Exam
June 9, 2022
Solutions

1. *Let A, B, and C be sets.*

   Prove that $A \cup C \subseteq B \cup C$ iff $A \setminus C \subseteq B \setminus C$.

   Suppose $A \cup B \subseteq B \cup C$.

   Suppose $x \in A \setminus C$.

   Then $x \in A$ and $x \notin C$.

   So $x \in A \cup C$ so $x \in B \cup C$ since $A \cup C \subseteq B \cup C$.

   Since $x \notin C$, $x \in B$.

   Hence, $x \in B \setminus C$.

   Thus, $x \in A \setminus C$ implies $x \in B \setminus C$, so $A \setminus C \subseteq B \setminus C$.

   Suppose $A \setminus C \subseteq B \setminus C$.

   Suppose $x \in A \cup C$.

   Suppose $x \in C$.

   Then $x \in B \cup C$.

   Suppose $x \notin C$.

   Then $x \in A$ and $x \notin C$, so $x \in A \setminus C$.

   Hence, $x \in B \setminus C$, since $A \setminus C \subseteq B \setminus C$.

   So, $x \in B$, and so $x \in B \cup C$.

   Hence, $x \in A \cup C$ implies $x \in B \cup C$, so $A \cup C \subseteq B \cup C$.

   Therefore, $A \cup C \subseteq B \cup C$ iff $A \setminus C \subseteq B \setminus C$.$\blacksquare$

2. *Let A be the set of all functions $f : \mathbb{R} \to \mathbb{R}$. Define a relation R on A by*

$$(f, g) \in R \text{ iff there exists a } c \in \mathbb{R} \text{ such that } f(x) = g(x) + c \text{ for all } x \in \mathbb{R}.$$

Prove that R is an equivalence relation.

Let $f \in A$.
Then $f(x) = f(x) + 0$ for all $x \in \mathbb{R}$.
Hence, $(f, f) \in R$.
Thus, R is reflexive.

Suppose $(f, g) \in R$.
Then there exists $c \in \mathbb{R}$ such that $f(x) = g(x) + c$ for all $x \in \mathbb{R}$.
Then $g(x) = f(x) + (-c)$ for all $x \in \mathbb{R}$.
Hence, $(g, f) \in R$ since $-c \in \mathbb{R}$.
Thus, R is symmetric.

Suppose $(f, g) \in R$ and $(g, h) \in R$.
Then there exists $c \in \mathbb{R}$ such that $f(x) = g(x) + c$ for all $x \in \mathbb{R}$.
Also, there exists $d \in \mathbb{R}$ such that $g(x) = h(x) + c$ for all $x \in \mathbb{R}$.
Then, $f(x) = h(x) + (c + d)$ for all $x \in \mathbb{R}$.
Hence, $(f, h) \in R$ since $c + d \in \mathbb{R}$.
Thus, R is transitive.

Since R is reflexive, symmetric and transitive, R is an equivalence relation.■

3. (a) *Let $\mathcal{F}$ and $\mathcal{G}$ be families. Prove that $\bigcup \mathcal{F} \setminus \bigcup \mathcal{G} \subseteq \bigcup(\mathcal{F} \setminus \mathcal{G})$.*

Suppose $x \in \bigcup \mathcal{F} \setminus \bigcup \mathcal{G}$.

Then $x \in \bigcup \mathcal{F}$ and $x \notin \bigcup \mathcal{G}$.

Hence, there exists $S \in \mathcal{F}$ such that $x \in S$.

Also, $S \notin G$ since $x \notin \bigcup \mathcal{G}$.

Hence, $S \in \mathcal{F} \setminus \mathcal{G}$, and thus $x \in \bigcup (\mathcal{F} \setminus \mathcal{G})$.

Thus, $x \in \bigcup \mathcal{F} \setminus \bigcup \mathcal{G}$ implies $x \in \bigcup (\mathcal{F} \setminus \mathcal{G})$.

Therefore, $\bigcup \mathcal{F} \setminus \bigcup \mathcal{G} \subseteq \bigcup(\mathcal{F} \setminus \mathcal{G})$.■

(b) *Prove that there exist non-empty families $\mathcal{F}$ and $\mathcal{G}$ such that $\bigcup \mathcal{F} \setminus \bigcup \mathcal{G} \neq \bigcup(\mathcal{F} \setminus \mathcal{G})$.*

Let $\mathcal{F} = \{\{1, 2\}\}$ and $\mathcal{G} = \{\{1\}\}$.

Then $\bigcup \mathcal{F} = \{1, 2\}$ and $\bigcup \mathcal{G} = \{1\}$.

So, $\bigcup \mathcal{F} \setminus \bigcup \mathcal{G} = \{2\}$.

Also, $\mathcal{F} \setminus \mathcal{G} = \mathcal{F}$ so

$$\bigcup(\mathcal{F} \setminus \mathcal{G}) = \{1, 2\} \neq \{2\} = \bigcup \mathcal{F} \setminus \bigcup \mathcal{G}.$$

Hence, there exist non-empty families $\mathcal{F}$ and $\mathcal{G}$ such that $\bigcup \mathcal{F} \setminus \bigcup \mathcal{G} \neq \bigcup(\mathcal{F} \setminus \mathcal{G})$. ■

4. *Let $f : \mathbb{R} \to \mathbb{R}$ be defined by*

$$f(x) = \begin{cases} x^2 - 2x + 3 & \text{if } x \leq 1; \\ -3x + 5 & \text{if } x > 1. \end{cases}$$

Prove that f is a bijection.

We first note that $x^2 - 2x + 3 = (x-1)^2 + 2$.

Since $(x-1)^2 \geq 0$ for all $x \in \mathbb{R}$, we conclude that $x^2 - 2x + 3 \geq 2$ for all $x \in \mathbb{R}$.

Hence $f(x) \geq 2$ for all $x \leq 1$.

Also, if $x > 1$, then $-3x < -3$ so $-3x + 5 < 2$, i.e., $f(x) < 2$.

Let $y \in \mathbb{R}$.

Suppose $y \geq 2$.

Let $a = 1 - \sqrt{y-2}$.

Then $a \leq 1$, so $f(a) = (a-1)^2 + 2 = y - 2 + 2 = y$.

Suppose $y < 2$.

Let $a = -\frac{1}{3}(y-5)$.

Since $y < 2$, $y - 5 < -3$ and $a = -\frac{1}{3}(y-5) > 1$ and so $f(a) = -3a + 5 = y - 5 + 5 = y$.

Thus, for all $y \in \mathbb{R}$, there exists $a \in \mathbb{R}$ such that $f(a) = y$, i.e., f is surjective.

Suppose $f(x_1) = f(x_2)$ for some $x_1, x_2 \in \mathbb{R}$.

Suppose $f(x_1) < 2$.

Then $f(x_2) < 2$ and so $f(x_1) = -3x_1 + 5 = -3x_2 + 5$.

Solving $-3x_1 + 5 = -3x_2 + 5$ we find $x_1 = x_2$.

Suppose $f(x_1) \geq 2$.

Then $f(x_2) \geq 2$ and $f(x_1) = (x_1 - 1)^2 + 1 = (x_2 - 1)^2 + 1$ and $x_1 \leq 1$ and $x_2 \leq 1$.

Hence, $(x_1 - 1)^2 = (x_2 - 1)^2$, so $|x_1 - 1| = |x_2 - 1|$.

Since $x_1 \leq 1$ and $x_2 \leq 1$, $|x_1 - 1| = 1 - x_1$ and $|x_2 - 1| = 1 - x_2$.

So we conclude $1 - x_1 = 1 - x_2$ and thus $x_1 = x_2$.

Thus $f(x_1) = f(x_2)$ implies $x_1 = x_2$: f is injective.

Since f is injective and surjective, f is a bijection. ■

5. *Prove that* $20! + 13$ *cannot be written as* $i^2 + j^4$ *for any integers* i *and* j*. (Hint: consider the situation modulo* 5*).*

By Euclid's Division theorem, every integer is congruent to 0, 1, 2, 3 or 4 modulo 5.

Considering squares and fourth powers modulo 5, we have these possibilities (note we can use the fact that $n^4 = (n^2)^2$ here.)

n	$n^2 (\text{mod} 5)$	$n^4 (\text{mod} 5)$
0	0	0
1	1	1
2	4	$16 \equiv 1$
3	$9 \equiv 4$	1
4	1	1

Thus, for any integer n, $n^2 \equiv 0, 1$ or 4 (mod 5) and $n^4 \equiv 0$ or 1 (mod 5).

We can calculate all possible sums modulo 5 of a square and a fourth power in this modulo 5 addition table:

$+$	0	1	4
0	0	1	4
1	1	2	0

Thus, we see that, for any integers i and j, $i^2 + j^4 \equiv 0, 1, 2$ or 4 (mod 5).

So $i^2 + j^4$ cannot be congruent to 3 (mod 5) for any integers i and j.

Now, $20! + 13 = 20(19!) + 10 + 3 = 5(4(19!) + 2) + 3 \equiv 3$ (mod 5).

Suppose $20! + 13 = i^2 + j^4$ for some integers i and j.

Then $i^2 + j^4 \equiv 3$ (mod 5).

This is a contradiction to our earlier statement that such a sum cannot be congruent to 3 modulo 5.

Hence, there do not exist integers i and j such that $i^2 + j^4 = 20! + 13$. ■

6. *Suppose A, B and C are sets. Suppose $f : A \to B$ and $g : B \to C$.*

   (a) *Prove that if f is onto and g is not one-to-one, then $g \circ f$ is not one-to-one.*

   Suppose f is onto and g is not one-to-one.

   Then there exist $x_1, x_2 \in B$ such that $g(x_1) = g(x_2)$ and $x_1 \neq x_2$.

   Since f is onto, there exist $a_1, a_2 \in A$ such that $f(a_1) = x_1$ and $f(a_2) = x_2$.

   Since $f(a_1) \neq f(a_2)$, we conclude $a_1 \neq a_2$.

   Then $g(f(a_1)) = g(f(a_2))$ with $a_1 \neq a_2$, so g is not one-to-one. ■

   (b) *Give an example to show that $g \circ f$ may be one-to-one when g is not one-to-one.*

   Let $A = \{a\}, B = \{b_1, b_2\}, C = \{y\}$.

   Let $f = \{(a, b_1)\}$ and $g = \{(b_1, y), (b_2, y)\}$.

   Then g is not one-to-one: $g(b_1) = g(b_2)$ but $b_1 \neq b_2$.

   Also, $g \circ f = \{(a, c)\}$, so g is one-to-one. ■

Elementary Properties of the Integers (EPIs)

Suppose a, b, c, and d are integers.

1. **Closure:** $a+b$ and ab are integers.
2. **Substitution of Equals:** If $a=b$, then $a+c=b+c$ and $ac=bc$.
3. **Commutativity:** $a+b=b+a$ and $ab=ba$.
4. **Associativity:** $(a+b)+c=a+(b+c)$ and $(ab)c=a(bc)$.
5. **The Distributive Law:** $a(b+c)=ab+ac$
6. **Identities:** $a+0=0+a=a$ and $a\cdot 1=1\cdot a=a$.

   0 is called the additive identity.

   1 is called the multiplicative identity.
7. **Additive Inverses:** There exists an integer $-a$ such that $a+(-a)=(-a)+a=0$.
8. **Trichotomy:** Exactly one of the following is true:
   $a>0$, $-a>0$, or $a=0$.
9. **The Well-Ordering Principle:** Every non-empty set of positive integers contains a smallest element.
10. $a\cdot 0=0$
11. If $a+c=b+c$, then $a=b$.
12. $-a=(-1)\cdot a$
13. $(-a)\cdot b=-(ab)$
14. $(-a)\cdot(-b)=ab$
15. If $ab=0$, then $a=0$ or $b=0$.
16. If $a\leq b$ and $b\leq a$, then $a=b$.
17. If $a<b$ and $b<c$, then $a<c$.
18. If $a<b$, then $a+c<b+c$.
19. If $a<b$ and $0<c$, then $ac<bc$.
20. If $a<b$ and $c<0$, then $bc<ac$.
21. If $a<b$ and $c<d$, then $a+c<b+d$.
22. If $0\leq a<b$ and $0\leq c<d$, then $ac<bd$.
23. If $a<b$, then $-b<-a$.
24. $0\leq a^2$, where $a^2=a\cdot a$.
25. If $ab=1$, then either $a=b=1$ or $a=b=-1$.

NOTE: Properties 17-23 hold if each $<$ is replaced with $\leq$.

One theorem for reference:
Theorem DAS (Divisors are Smaller):
Let a and b be positive integers. Then $a|b$ implies $a\leq b$.

Sets

$A\subseteq B$ iff $x\in A$ implies $x\in B$

$A=B$ iff $A\subseteq B$ and $B\subseteq A$

$x\in A\cup B$ iff $x\in A$ or $x\in B$

$x\in A\cap B$ iff $x\in A$ and $x\in B$

$x\in A\setminus B$ iff $x\in A$ and $x\notin B$