Math 300 B - Winter 2015
Final Exam
March 18, 2015
Answers

1. *Let a and b be integers. Prove that $3|a^2+b^2$ if and only if $3|a$ and $3|b$.*

   Let a and b be integers.

   By Euclid's theorem, $a = 3k + r$ and $b = 3m + s$ for integers k, m, r and s, with $0 \leq r \leq 2$ and $0 \leq s \leq 2$.

   Then

$$\begin{aligned} a^2 + b^2 = (3k+r)^2 + (3m+s)^2 &= 9k^2 + 6kr + r^2 + 9m^2 + 6ms + s^2 \\ &= 3(3k^2 + 2kr + 3m^2 + 2ms) + r^2 + s^2. \end{aligned}$$

   Suppose $a^2 + b^2 = 3t$ for some integer t.

   Then $r^2 + s^2 = 3(t - (3k^2 + 2kr + 3m^2 + 2ms))$ and so $r^2 + s^2$ is divisible by 3.

   On the other hand, suppose $r^2 + s^2 = 3v$ for some integer v. Then

$$a^2 + b^2 = 3((3k^2 + 2kr + 3m^2 + 2ms) + v),$$

   and so $a^2 + b^2$ is divisible by 3.

   Thus, $a^2 + b^2$ is divisible by 3 if and only if $r^2 + s^2$ is divisible by 3.

   We have the following table of possible value of $r^2 + s^2$.

	0	1	2
0	0	1	4
1	1	2	5
2	2	5	8

   Since $1, 2, 4, 5$ and 8 are not divisible by 3, we conclude that $a^2 + b^2$ is divisible by 3 if and only if $r = s = 0$; that is, if and only if $3|a$ and $3|b$. ■

2. *Let A and B be sets. Prove that*

$$A \subseteq B$$

if and only if

$$\mathcal{P}(A) \subseteq \mathcal{P}(B).$$

Let A and B be sets.

Suppose $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

Note that $A \in \mathcal{P}(A)$, so $A \in \mathcal{P}(B)$.

Hence $A \subseteq B$.

Suppose $A \subseteq B$.

Let $x \in \mathcal{P}(A)$.

Then $x \subseteq A$.

Suppose $y \in x$.

Then $y \in A$, and so $y \in B$.

Thus $y \in x$ implies $y \in B$, and hence $x \subset B$.

Thus, $x \in \mathcal{P}(B)$.

Hencer, $x \in \mathcal{P}(A)$ implies $x \in \mathcal{P}(B)$, so $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.$\blacksquare$

3. *Use induction to prove that*

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$$

for all $n \in \mathbb{Z}_{>0}$.

Let $n \in \mathbb{Z}_{>0}$.

Let $P(n)$ be the statement "$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$".

Base Case: Let $n = 1$.

Then

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{1}{1(1+1)} = \frac{1}{2} = \frac{1}{1+1} = \frac{n}{n+1}.$$

Thus, $P(1)$ is true.

Induction Step: Suppose $P(n)$ is true for some $n = k > 0$.

So $P(k)$ is true. That is,

$$\sum_{i=1}^{k} \frac{1}{i(i+1)} = \frac{k}{k+1}$$

Then

$$\begin{aligned}
\sum_{i=1}^{k+1} \frac{1}{i(i+1)} &= \sum_{i=1}^{k} \frac{1}{i(i+1)} + \frac{1}{(k+1)(k+2)} \\
&= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} \\
&= \frac{k(k+2)}{(k+1)(k+2)} + \frac{1}{(k+1)(k+2)} \\
&= \frac{k(k+2)+1}{(k+1)(k+2)} \\
&= \frac{k^2+2k+1}{(k+1)(k+2)} \\
&= \frac{(k+1)^2}{(k+1)(k+2))} \\
&= \frac{k+1}{k+2}
\end{aligned}$$

Hence $P(k+1)$ is true.

Thus, $P(k)$ implies $P(k+1)$, and $P(1)$ is true, so, by induction, $P(n)$ is true for all $n \in \mathbb{Z}_{>0}$.■

4. *Suppose A, B and C are sets with $A \subseteq C$ and $B \subseteq C$. Suppose $A \cap B = \varnothing$, and that $A \sim B$. Prove that*
$$C \setminus A \sim C \setminus B.$$

Suppose A, B and C are sets with $A \subseteq C$ and $B \subseteq C$. Suppose $A \cap B = \varnothing$, and that $A \sim B$.

Then there exists a bijection $f : A \to B$.

Note that since f is a bijection, $f^{-1} : B \to A$ is a bijection.

Define a function $g : C \setminus A \to C \setminus B$ by

$$g(x) = \begin{cases} x & \text{if } x \notin B, \\ f^{-1}(x) & \text{if } x \in B. \end{cases}$$

We prove that g is a bijection.

Suppose $y \in C \setminus B$.

Suppose $y \in A$. Then $f(y) \in B$.

Since $A \cap B = \varnothing$, $f(y) \in C \setminus A$, and we have $g(f(y)) = f^{-1}(f(y)) = y$.

Suppose $y \notin A$. Then $y \notin B$, so $g(y) = y$.

Hence, g is surjective.

Suppose $g(x_1) = g(x_2)$.

Suppose $x_1, x_2 \in B$. Then
$$f^{-1}(x_1) = f^{-1}(x_2)$$
and so $x_1 = x_2$.

Suppose $x_1, x_2 \notin B$.

Then $g(x_1) = x_1 = g(x_2) = x_2$.

Suppose exactly one of x_1, x_2 is in B.

Without loss of generality, suppose $x_1 \in B$ and $x_2 \notin B$.

Then $g(x_1) = f^{-1}(x_1) = g(x_2) = x_2$.

Since $x_2 \in C \setminus A$, $x_2 \notin A$.

However, $f^{-1}(x_1) \in A$, and this contradicts the statement $g(x_1) = g(x_2)$.

Thus, g is injective.

Hence, g is a bijection.■

5. *Suppose A, B and C are sets. Suppose $f : A \to B$ and $g : B \to C$.*

(a) *Prove that if f is onto and g is not one-to-one, then $g \circ f$ is not one-to-one.*

Suppose A, B and C are sets. Suppose $f : A \to B$ and $g : B \to C$, f is onto and g is not one-to-one.

Since g is not one-to-one, there exist $b_1 \neq b_2 \in B$ such that $g(b_1) = g(b_2)$.

Since f is onto, there exist a_1 and a_2 such that $f(a_1) = b_1$ and $f(a_2) = b_2$.

Note that $a_1 \neq a_2$ since $b_1 \neq b_2$.

Then $g(f(a_1)) = g(b_1) = g(b_2) = g(f(a_2))$, so $g \circ f$ is not one-to-one.■

(b) *Prove that if f is not onto and g is one-to-one, then $g \circ f$ is not onto.*

Suppose A, B and C are sets. Suppose $f : A \to B$ and $g : B \to C$, f is not onto and g is one-to-one.

Since f is not onto, there is a $b \in B$ such that $f(a) \neq b$ for all $a \in A$.

Suppose $g(f(a)) = g(b)$ for some $a \in A$.

Then, since g is one-to-one, $f(a) = b$.

This is a contradiction: for all $a \in A$, $f(a) \neq b$.

Thus, for all $a \in A$, $g(f(a)) \neq g(b)$.

Hence, $g \circ f$ is not onto. ■

6. *Let $A = \mathbb{R} \times \mathbb{R} \setminus \{(0,0)\}$.*

   Thus, A is the xy-plane without the origin.

   Define a relation R on A by

   $((x_1, y_1), (x_2, y_2)) \in R \Leftrightarrow (x_1, y_1)$ and (x_2, y_2) lie on a line which passes through the origin.

   Prove that R is an equivalence relation.

   Is R reflexive?

   Suppose $P \in A$. Then $P = (x_1, y_1)$ for some $x_1, y_1 \in \mathbb{R}$.

   Suppose $x_1 = 0$. Then P lies on the line $x = 0$ which passes through the origin.

   Hence, $(P, P) \in R$.

   Suppose $x_1 \neq 0$. Then P lies on the line $y = \frac{y_1}{x_1}x$, which passes through the origin.

   Hence, $(P, P) \in R$.

   Thus, in all cases, $(P, P) \in R$, so R is reflexive.

   Is R symmetric?

   Suppose $P = (x_1, y_1) \in A$ and $Q = (x_2, y_2) \in A$ and $(P, Q) \in R$.

   Then P and Q lie on a line through the origin, and so Q and P lie on a line through the origin.

   Hence, $(Q, P) \in R$, and so R is symmetric.

   Is R transitive?

   Suppose $P = (x_1, y_1) \in A$ and $Q = (x_2, y_2) \in A$ and $S = (x_3, y_3) \in A$ and $(P, Q) \in R$ and $(Q, S) \in R$.

   Suppose $x_1 = 0$.

   Then P lies on the vertical line $x = 0$ through the origin and no other line through the origin.

   Hence, Q lies on $x = 0$, and hence S lies on $x = 0$.

   Thus, P and S lie on a line through the origin, and so $(P, S) \in R$.

   Suppose $x_1 \neq 0$.

   Then P and Q lie on a line $y = mx$ where $m \in \mathbb{R}$, and Q and R lie on a line $y = nx$ where $n \in \mathbb{R}$.

   Then $y_2 = mx_2 = nx_2$ so $(m - n)x_2 = 0$.

   If $x_2 = 0$, then $y_2 = mx_2 = 0$, so $Q = (0, 0) \notin A$, a contradiction since $Q \in A$.

   Hence $x_2 \neq 0$, so $m - n = 0$, i.e., $m = n$.

   Thus, P and Q lie on the line $y = mx$ and Q and S lie on the line $y = mx$, so P and S both lie on a line through the origin.

   Hence, $(P, S)\ inR$ and thus R is transitive.

   Since R is reflexive, symmetric, and transitive, R is an equivalence relation. ■