Math 300 E - Winter 2019
Final Exam
March 18, 2019

Name: ______________________ Student ID no. : ______________

Signature: ____

1	10	
2	10	
3	10	
4	10	
5	9	
6	10	
Total	59	

- You have 110 minutes to complete the exam.

1. Let A, B, and C be sets. Suppose $f : A \to B$ and $g : B \to C$.

   (a) Prove that, if $g \circ f$ is one-to-one, then f is one-to-one.

   (b) Give an example to show that, if $g \circ f$ is one-to-one, g need not be one-to-one.

2. Use induction to prove that $2^n > n^2$ for all integers $n \geq 5$.

3. (a) Let m be a positive integer.
Suppose a, b, c and d are integers and $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$.
Prove that $ac \equiv bd \pmod{m}$.

(b) Prove that $11 \mid 4^{1234} - 3$.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \frac{1}{2}x + 3 & \text{if } x \geq 0 \\ 5x + 3 & \text{if } x < 0. \end{cases}$$

Prove that f is a bijection from $\mathbb{R}$ to $\mathbb{R}$.

5. Let $A = \{a, b, c\}$. Give an example of each of the following.

   (a) An equivalence relation on A.

   (b) A relation on A that is reflexive but not symmetric.

   (c) A relation on A that is a function whose inverse is not a function.

6. Let A and B be sets. Prove that $A \cap B = \varnothing$ if and only if $\mathcal{P}(A) \cap \mathcal{P}(B) = \{\varnothing\}$.

Axioms of the Integers (AIs)

Suppose a, b, and c are integers.

- **Closure:**

  $a + b$ and ab are integers.

- **Substitution of Equals:**

  If $a = b$, then $a+c = b+c$ and $ac = bc$.

- **Commutativity:**

  $a + b = b + a$ and $ab = ba$.

- **Associativity:**

  $(a + b) + c = a + (b + c)$ and $(ab)c = a(bc)$.

- **The Distributive Law:**

  $a(b + c) = ab + ac$

- **Identities:**

  $a + 0 = 0 + a = a$ and $a \cdot 1 = 1 \cdot a = a$

  0 is called the *additive identity*

  1 is called the *multiplicative identity*.

- **Additive Inverses:**

  There exists an integer $-a$ such that $a + (-a) = (-a) + a = 0$.

- **Trichotomy:**

  Exactly one of the following is true: $a < 0$, $-a < 0$, or $a = 0$.

Sets

$A \subseteq B$ iff $x \in A$ implies $x \in B$

$A = B$ iff $A \subseteq B$ and $B \subseteq A$

$x \in A \cup B$ iff $x \in A$ or $x \in B$

$x \in A \cap B$ iff $x \in A$ and $x \in B$

$x \in A \setminus B$ iff $x \in A$ and $x \notin B$

$\mathcal{P}(A)$ is the set of all subsets of a set A

Elementary Properties of the Integers (EPIs)

Suppose a, b, c, and d are integers.

1. $a \cdot 0 = 0$
2. If $a + c = b + c$, then $a = b$.
3. $-a = (-1) \cdot a$
4. $(-a) \cdot b = -(a \cdot b)$
5. $(-a) \cdot (-b) = a \cdot b$
6. If $a \cdot b = 0$, then $a = 0$ or $b = 0$.
7. If $a \leq b$ and $b \leq a$, then $a = b$.
8. If $a < b$ and $b < c$, then $a < c$.
9. If $a < b$, then $a + c < b + c$.
10. If $a < b$ and $0 < c$, then $ac < bc$.
11. If $a < b$ and $c < 0$, then $bc < ac$.
12. If $a < b$ and $c < d$, then $a + c < b + d$.
13. If $0 \leq a < b$ and $0 \leq c < d$, then $ac < bd$.
14. If $a < b$, then $-b < -a$.
15. $0 \leq a^2$
16. If $ab = 1$, then either $a = b = 1$ or $a = b = -1$.

NOTE: Properties 8-14 hold if each $<$ is replaced with $\leq$.

One theorem for reference:

Theorem DAS (Divisors are Smaller): Let a and b be positive integers. Then $a|b$ implies $a \leq b$.