Math 300 D - Autumn 2014
Midterm Exam Number Two
November 12, 2014
Answers

1. *Let a and b be integers. Prove that $x = a^2 + ab + b$ is odd iff a is odd or b is odd.*

   Suppose a and b are integers.

   We can consider four cases.

   (a) Suppose a is even and b is even.
   Then $a = 2k$ and $b = 2m$ for some integers k and m.
   Then $x = 4k^2 + 4km + 2m = 2(2k^2 + 2km + m)$.
   Since $2k^2 + 2km + m$ is an integer (by Axiom 11), x is even.

   (b) Suppose a is even and b is odd.
   Then $a = 2k$ and $b = 2m + 1$ for some integers k and m.
   Then $x = 4k^2 + 2k(2m + 1) + 2m + 1 = 2(2k^2 + k(2m + 1) + m) + 1$.
   Since $2k^2 + k(2m + 1) + m$ is an integer (by Axiom 11), x is odd.

   (c) Suppose a is odd and b is even.
   Then $a = 2k + 1$ and $b = 2m$ for some integers k and m.
   Then

   $$\begin{aligned} x &= (2k + 1)^2 + (2k + 1)(2m) + 2m \\ &= 4k^2 + 4k + 1 + (2k + 1)(2m) + 2m \\ &= 2(2k^2 + 2k + m(2k + 1) + m) + 1. \end{aligned}$$

   Since $2k^2 + 2k + m(2k + 1) + m$ is an integer (by Axiom 11), x is odd.

   (d) Suppose a is odd and b is odd.
   Then $a = 2k + 1$ and $b = 2m+!$ for some integers k and m.
   Then

   $$\begin{aligned} x &= (2k + 1)^2 + (2m + 1)(2k + 1) + b \\ &= 4k^2 + 4k + 1 + 4mk + 2m + 2k + 1 + 2m + 1 \\ &= 2(2k^2 + 2mk + 3k + 2m + 1) + 1. \end{aligned}$$

   Since $(2k^2 + 2mk + 3k + 2m + 1)$ is an integer (by Axiom 11), x is odd.

   The first case shows that if a and b are even, then x is even; hence, if x is odd, then a is odd or b is odd.

   The other three cases show that if a or b is odd, then x is odd.

   Hence, x is odd if and only if a is odd or b is odd. ■

2. *Let $S = \mathbb{R}_{>0} \times \mathbb{R}_{>0}$. Define a relation $R \subseteq S \times S$ by*

$$((x_1, y_1), (x_2, y_2)) \in R \Leftrightarrow x_1 y_1 = x_2 y_2.$$

(a) *Prove that R is an equivalence relation.*

Suppose $P = (x, y) \in S \times S$.
Since $xy = xy$, $((x, y), (x, y) \in R$, i.e., $(P, P) \in R$.
Hence, R is reflexive.

Suppose $((x_1, y_1), (x_2, y_2)) \in R$.
Then $x_1 y_1 = x_2 y_2$.
Hence, $x_2 y_2 = x_1 y_1$, so $((x_2, y_2), (x_1, y_1)) \in R$.
Thus, R is symmetric.

Suppose $((x_1, y_1), (x_2, y_2)) \in R$ and $((x_2, y_2), (x_3, y_3)) \in R$.
Then $x_1 y_1 = x_2 y_2$ and $x_2 y_2 = x_3 y_3$.
Hence, $x_1 y_1 = x_3 y_3$.
Thus, $((x_1, y_1), (x_3, y_3)) \in R$.
Hence, R is transitive.

Since R is reflexive, symmetric and transitive, R is an equivalence relation.■

(b) *Notice that elements of S can be viewed as points in the first quadrant of the xy-plane (i.e., the set of points (x, y) where $x > 0$ and $y > 0$.) Draw a picture of one equivalence class in S/R and indicate which equivalence class it is.*

Consider $(1, 1) \in S$.
We have $((1, 1), (x, y)) \in R$ iff $1 = xy$, i.e., $y = 1/x$.
Thus, the portion of the curve $y = 1/x$ in the first quadrant is the equivalence class $[(1, 1)]$ in S/R.
(In fact, for any positive j, the curve $y = j/x$ represents an equivalance class in S/R.)

3. *Let A and B be sets. Prove that $A = B$ iff $\mathcal{P}(A) = \mathcal{P}(B)$.*

Let A and B be sets.

Suppose $A = B$. Then $\mathcal{P}(A) = \mathcal{P}(B)$.

Suppose $\mathcal{P}(A) = \mathcal{P}(B)$.

Suppose $x \in A$.

Then $\{x\} \in \mathcal{P}(A)$, so $\{x\} \in \mathcal{P}(B)$.

Hence, $\{x\} \subseteq B$.

Since $x \in \{x\}$, $x \in B$.

Hence, since $x \in A$ implies $x \in B$, $A \subseteq B$.

Now, suppose $y \in B$.

Then $\{y\} \in \mathcal{P}(B)$, so $y \in \mathcal{P}(A)$.

Hence, $\{y\} \subseteq A$.

Since $y \in \{y\}$, $y \in A$.

Hence, since $y \in B$ implies $y \in A$, $B \subseteq A$.

Thus, $A \subseteq B$ and $B \subseteq A$, so $A = B$.

Therefore, $A = B$ iff $\mathcal{P}(A) = \mathcal{P}(B)$. ■

4. *Prove that* $\sqrt{2} + \sqrt{3}$ *is an algebraic number.*

   Let $x = \sqrt{2} + \sqrt{3}$.

   Then $(x - \sqrt{2})^2 = 3$

   so $x^2 - 2\sqrt{2}x - 1 = 0$.

   Hence, $(-2\sqrt{2}x) = (1 - x^2)^2$, i.e.

   $8x^2 = 1 - 2x^2 + x^4$, i.e.

   $0 = x^4 - 10x^2 + 1$.

   This shows that x is a root of a polynomial with integer coefficients, and so x is an algebraic number.■

5. *Let* $\mathcal{F}$ *be a family of sets, and* B *be a set. Prove that if* $\bigcup \mathcal{F} \subseteq B$, *then* $\mathcal{F} \subseteq \mathcal{P}(B)$.

   Let $\mathcal{F}$ be a family of sets, and B be a set.

   Suppose $\bigcup \mathcal{F} \subseteq B$.

   Let $x \in \mathcal{F}$.

   Let $y \in x$.

   Then $y \in \bigcup \mathcal{F}$.

   Hence, $y \in B$.

   Since $y \in x$ implies $y \in B$, $x \subseteq B$.

   Hence, $x \in \mathcal{P}(B)$.

   Since $x \in \mathcal{F}$ implies $x \in \mathcal{P}(B)$, $\mathcal{F} \subseteq \mathcal{P}(B)$. ■