Math 300 A, B - Spring 2012
Midterm Exam
April 27, 2012
Answers

1. (Short answer)

   (a) Let A be the set $\{a, b\}$. List every element of the power set of A.

   $$\emptyset, \{a\}, \{b\}, \{a, b\}$$

   (b) Use truth tables to show that $\neg P \vee Q$ and $\neg(P \wedge \neg Q)$ are equivalent.

P	Q	$\neg P$	$\neg P \vee Q$	$\neg Q$	$P \wedge \neg Q$	$\neg(P \wedge \neg Q)$
F	F	T	T	T	F	T
F	T	T	T	F	F	T
T	F	F	F	T	T	F
T	T	F	T	F	F	T

   (c) Give a useful negation of this statement:
   "For every integer n, there is an integer m such that $m|n$."
   One negation is: "There exists an integer n, such that, for all integers m, m does not divide n".

2. *Prove that, for any sets A and B,*

   $$\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B).$$

   Proof. Suppose $x \in \mathcal{P}(A) \cup \mathcal{P}(B)$. Then $x \in \mathcal{P}(A)$ or $x \in \mathcal{P}(B)$.
   Suppose $x \in \mathcal{P}(A)$.
   Then $x \subseteq A$.
   Suppose $y \in x$. Then $y \in A$, so $y \in A \cup B$. Hence, $x \subseteq A \cup B$.
   Therefore, $x \in \mathcal{P}(A \cup B)$.
   An identical argument shows that if $x \in \mathcal{P}(B)$, then $x \in \mathcal{P}(A \cup B)$.
   Thus, if $x \in \mathcal{P}(A) \cup \mathcal{P}(B)$, then $x \in \mathcal{P}(A \cup B)$.
   Therefore, $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$. $\square$

3. *Prove that, for all integers n, $n^2 - 2$ is not divisible by 3. (HINT: Every integer can be written in the form $3k + r$, for k and r integers, and $r = 0, 1,$ or 2).*

Proof. Let n be an integer. Then $n = 3k + r$ for integers k and r with $r = 0, 1$ or 2.

Suppose $r = 0$. Then $n = 3k$, so $n^2 - 2 = 9k^2 - 2$ and

$$\frac{n^2 - 2}{3} = 3k^2 - \frac{2}{3}$$

which is not an integer. So 3 does not divide $n^2 - 2$.

Suppose $r = 1$. Then $n = 3k + 1$, so $n^2 - 2 = 9k^2 + 6k - 1$ and

$$\frac{n^2 - 2}{3} = 3k^2 + 2k - \frac{1}{3}$$

which is not an integer. So 3 does not divide $n^2 - 2$.

Suppose $r = 2$. Then $n = 3k + 2$, so $n^2 - 2 = 9k^2 + 12k + 2$ and

$$\frac{n^2 - 2}{3} = 3k^2 + 4k + \frac{2}{3}$$

which is not an integer. So 3 does not divide $n^2 - 2$.

Thus, 3 does not divide $n^2 - 2$ for any integer n. □

4. *Let n be an integer. Prove that $20|n$ iff $4|n$ and $5|n$.*

Proof. Let n be an integer.

Suppose $20|n$. Then $n = 20k$ for some integer k. Hence $n = 4(5k)$ and $n = 5(4k)$ so $4|n$ and $5|n$.

Suppose $4|n$ and $5|n$. Then $n = 4k$ and $n = 5m$ for integers k and m. Hence,

$$n = 5n - 4n = 20k - 20m = 20(k - m)$$

and, since $k - m$ is an integer, $20|n$.

Thus, $20|n$ iff $4|n$ and $5|n$. □

5. *Prove that for all $x \in \mathbb{R}$,$x \neq -\frac{1}{2}$, there exists a $y \in \mathbb{R}$ such that $y \neq x$ and $y^2 - x = x^2 - y$.*

Proof. Let $x \in \mathbb{R}$ such that $x \neq -\frac{1}{2}$.

Let $y = -1 - x$.

Then

$$y^2 - x = (-1 - x)^2 - x = 1 + 2x + x^2 - x = 1 + x + x^2$$

and

$$x^2 - y = x^2 - (1 - x) = x^2 + x + 1$$

so $y^2 - x = x^2 - y$.

Suppose $y = x$. Then $-1 - x = x$ and so $x = -\frac{1}{2}$.

This is a contradiction to our assumption that $x \neq -\frac{1}{2}$. Hence, $y \neq x$. □