Math 300 B, C - Spring 2013
Midterm Exam
April 22, 2013
Answers

1. *For sets A, B, and C, show the following identities using logic symbols and equivalences.*

   (a) $A \setminus (A \cap B) = A \setminus B$
   We have

$$
\begin{aligned}
& x \in (A \setminus (A \cap B)) & \\
\Leftrightarrow\ & x \in A \wedge x \notin (A \cap B) & \text{(definition of set difference)} \\
\Leftrightarrow\ & x \in A \wedge \neg(x \in (A \cap B)) & \text{(definition of negation)} \\
\Leftrightarrow\ & x \in A \wedge \neg(x \in A \wedge x \in B) & \text{(definition of set intersection)} \\
\Leftrightarrow\ & x \in A \wedge (x \notin A \vee x \notin B) & \text{(DeMorgan's law)} \\
\Leftrightarrow\ & (x \in A \wedge x \notin A) \vee (x \in A \vee x \notin B) & \text{(distribution law)} \\
\Leftrightarrow\ & x \in A \wedge x \notin B & \text{(contradiction)} \\
\Leftrightarrow\ & x \in A \setminus B. & \text{(definition of set difference)}
\end{aligned}
$$

   Hence, $A \setminus (A \cap B) = A \setminus B$.

   (b) $(A \setminus B) \setminus C = A \setminus (B \cup C)$
   We have

$$
\begin{aligned}
& x \in (A \setminus B) \setminus C & \\
\Leftrightarrow\ & (x \in A \wedge x \notin B) \wedge (x \notin C) & \text{(definition of set difference)} \\
\Leftrightarrow\ & (x \in A) \wedge (x \notin B \wedge x \notin C) & \text{(associative law)} \\
\Leftrightarrow\ & (x \in A) \wedge \neg(x \in B \vee x \in C) & \text{(DeMorgan's law)} \\
\Leftrightarrow\ & (x \in A) \wedge x \notin (B \cup C) & \text{(definition of set union)} \\
\Leftrightarrow\ & x \in A \setminus (B \cup C) & \text{(definition of set difference).}
\end{aligned}
$$

   Hence, $(A \setminus B) \setminus C = A \setminus (B \cup C)$.

   (c) $(A \setminus B) \cap (B \setminus A) = \varnothing$
   We have

$$
\begin{aligned}
& x \in ((A \setminus B) \cap (B \setminus A)) & \\
\Leftrightarrow\ & x \in (A \setminus B) \wedge x \in (B \setminus A) & \text{(definition of set intersection)} \\
\Leftrightarrow\ & x \in A \wedge x \notin B \wedge x \in B \wedge x \notin A & \text{(definition of set difference)} \\
\Leftrightarrow\ & (x \in A \wedge x \notin A) \wedge (x \in B \wedge x \notin B) & \text{(associativity and commutativity)}
\end{aligned}
$$

Since $(x \in A \wedge x \notin A)$ and $(x \in B \wedge x \notin B)$ are contradictions, $x \in ((A \setminus B) \cap (B \setminus A))$ is a contradictions. Hence, $x \in ((A \setminus B) \cap (B \setminus A))$ is equivalent to all other contradictions; in particular, we have

$$x \in ((A \setminus B) \cap (B \setminus A)) \Leftrightarrow x \in \varnothing.$$

Hence $(A \setminus B) \cap (B \setminus A) = \varnothing$.

2. Write useful contrapositives of each of the following sentences. All variables represent integers.

   (a) *If $xy = 3$ and $x < y$, then $x = 1$ and $y = 3$.*
   The contrapositive is: "If $x \neq 1$ or $y \neq 3$, then $xy \neq 3$ or $x \geq y$."

   (b) *If x is even or y is odd, then $x(y-1)$ is even.*
   The contrapositive is: "If $x(y-1)$ is odd, then x is odd and y is even."

   (c) *If there exists a prime p such that p^2 divides x, then x is not squarefree. (Your contrapositive should incorporate a "for all" statement.)*
   The contrapositive is: "If x is squarefree, then for all primes, p, p^2 does not divide x."

3. Simplify the following expressions. Justify your results by showing a sequence of equivalent expressions connecting the original expression with your final one.

   (a) $(P \vee Q) \vee \neg(\neg P \vee \neg R)$
   We have:

$$\begin{aligned}
& (P \vee Q) \vee \neg(\neg P \vee \neg R) & \\
\Leftrightarrow\ & (P \vee Q) \vee (P \wedge R) & \text{(DeMorgan's law)} \\
\Leftrightarrow\ & (P \vee Q \vee P) \wedge (P \vee Q \vee R) & \text{(distributive law)} \\
\Leftrightarrow\ & (Q \vee P) \wedge (P \vee Q \vee R) & \text{(idempotent law)} \\
\Leftrightarrow\ & (Q \vee P) & \text{(absorption law)}
\end{aligned}$$

   (b) $(P \vee \neg(\neg P \wedge \neg Q)) \wedge \neg((\neg P \wedge R) \vee (R \wedge \neg R))$
   We have:

$$\begin{aligned}
& (P \vee \neg(\neg P \wedge \neg Q)) \wedge \neg((\neg P \wedge R) \vee (R \wedge \neg R)) & \\
\Leftrightarrow\ & (P \vee P \vee Q) \wedge (\neg((\neg P \wedge R) \vee (R \wedge \neg R)) & \text{(DeMorgan's law)} \\
\Leftrightarrow\ & (P \vee Q) \wedge ((P \vee \neg R) \wedge (\neg R \vee R)) & \text{(DeMorgan's law)} \\
\Leftrightarrow\ & (P \vee Q) \wedge (P \vee \neg R) & \text{(tautology)} \\
\Leftrightarrow\ & P \vee (Q \wedge \neg R) & \text{(distributive law)}
\end{aligned}$$

4. *Write a truth table for the statement $\neg P \wedge (Q \vee P)$*

   The following is a truth table for the statement $\neg P \wedge (Q \vee P)$.

P	Q	$\neg P$	$Q \vee P$	$\neg P \wedge (Q \vee P)$
F	F	T	F	F
F	T	T	T	T
T	F	F	T	F
T	T	F	T	F

5. *Find a formula using only* $\neg$ *and* $\wedge$ *that is equivalent to* $(P \to Q) \to \neg(Q \vee P)$.

One way is like this:

$$
\begin{aligned}
& (P \to Q) \to \neg(Q \vee P) & \\
\Leftrightarrow\ & (P \to Q) \to (\neg Q \wedge \neg P) & \text{(DeMorgan's law)} \\
\Leftrightarrow\ & (\neg P \vee Q) \to (\neg Q \wedge \neg P) & \text{(conditional law)} \\
\Leftrightarrow\ & \neg(P \wedge \neg Q) \to (\neg Q \wedge \neg P) & \text{(DeMorgan's law)} \\
\Leftrightarrow\ & (P \wedge \neg Q) \vee (\neg Q \wedge \neg P) & \text{(conditional law)} \\
\Leftrightarrow\ & \neg(\neg(P \wedge \neg Q) \wedge \neg(\neg Q \wedge \neg P)) & \text{(DeMorgan's law)}
\end{aligned}
$$