Math 300 C - Spring 2015
Midterm Exam Number One
April 22, 2015
Solutions

1. *Let A, B, and C be sets. Suppose $A \cup C \subseteq B \cup C$. Prove that $A \setminus C \subseteq B$.*

   Let A, B, and C be sets.

   Suppose $A \cup C \subseteq B \cup C$.

   Suppose $x \in A \setminus C$.

   Then $x \in A$ and $x \notin C$.

   Hence, $x \in A \cup C$.

   Since $A \cup C \subseteq B \cup C$, $x \in B \cup C$.

   Hence, $x \in B$ or $x \in C$.

   Since $x \notin C$, $x \in B$.

   Thus, $x \in A \setminus C$ implies $x \in B$.

   Therefore, $A \setminus C \subseteq B$. ■

2. *Let A, B and C be sets. Prove that, if $A \subseteq B \cup C$, then $A = (A \cap B) \cup (A \cap C)$.*

   Let A, B and C be sets.

   Suppose $A \subseteq B \cup C$.

   Suppose $x \in A$.

   Then $x \in B$ or $x \in C$.

   Suppose $x \in B$.

   Then $x \in A$ and $x \in B$, so $x \in A \cap B$.

   Hence, $x \in (A \cap B) \cup (A \cap C)$.

   Suppose $x \in C$.

   Then $x \in A$ and $x \in C$, so $x \in A \cap C$.

   Hence, $x \in (A \cap B) \cup (A \cap C)$.

   Thus, $x \in (A \cap B) \cup (A \cap C)$.

   Hence, $x \in A$ implies $x \in (A \cap B) \cup (A \cap C)$, and so $A \subseteq (A \cap B) \cup (A \cap c)$.

   Now, suppose $x \in (A \cap B) \cup (A \cap C)$.

   Then $x \in A \cap B$ or $A \in A \cap C$.

   Suppose $x \in A \cap B$.

   Then $x \in A$.

Suppose $x \in A \cap C$.

Then $x \in A$.

Thus $x \in A$, so $x \in (A \cap B) \cup (A \cap C)$ implies $x \in A$.

Therefore, $(A \cap B) \cup (A \cap C) \subseteq A$.

Thus, $A = (A \cap B) \cup (A \cap C)$. ■

3. *Let $a, b,$ and c be positive integers. Prove that, if $a|bc$ and $a|(b+c)$, then $a|b^2$.*

   Let $a, b,$ and c be positive integers.

   Suppose $a|bc$ and $a|(b+c)$.

   Then there exists an integer k such that $bc = ak$.

   Also, there exists an integer m such that $b + c = am$.

   Solving for b, we have $b = am - c$ (Additive Inverse Axiom).

   Multiplying both sides by b, we obtain $b^2 = (am - c)b = abm - bc$ (Substitution of Equals, Distributive Law, Commutativity).

   Hence, $b^2 = abm - ak = a(bm - k)$ (Substitution of Equals, Distributive Law).

   Since $bm - k \in \mathbb{Z}$ (since $\mathbb{Z}$ is closed under addition and multiplication), we conclude that $a|b^2$. ■

4. *Let $S = \{1, 2, 3\}$. Write out the set $\mathcal{P}(S)$ by listing its elements.*

$$\mathcal{P}(S) = \mathcal{P}(\{1,2,3\}) = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$$

5. *Suppose A and B are sets. Suppose $\mathcal{P}(A \setminus B) = \mathcal{P}(A)$. Prove that $A \cap B = \varnothing$.*

   We prove the contrapositive.

   Suppose A and B are sets.

   Suppose $A \cap B \neq \varnothing$.

   Then there exists an $x \in A \cap B$.

   So, $x \in A$ and $x \in B$.

   Hence, $\{x\} \subseteq A$, so $\{x\} \in \mathcal{P}(A)$.

   Suppose $\{x\} \in \mathcal{P}(A \setminus B)$.

   Then $\{x\} \subseteq A \setminus B$.

   Then $x \in A \setminus B$; that is, $x \in A$ and $x \notin B$.

   This is a contradiction, since $x \in B$.

   Thus, $\{x\} \notin \mathcal{P}(A \setminus B)$.

   Since $\{x\} \in \mathcal{P}(A)$, we conclude that $\mathcal{P}(A) \neq \mathcal{P}(A \setminus B)$.■