Math 300 C - Spring 2016
Midterm Exam Number One
April 20, 2016
Answers

1. **Theorem:** Suppose x and a are integers and $a > 0$.

   For any integer y, define $|y|$ by

$$|y| = \begin{cases} y & \text{if } y \geq 0, \\ -y & \text{if } y < 0. \end{cases}$$

   Then $|ax| = a|x|$.

   Proof: Suppose x and a are integers and $a > 0$.

   By the Trichotomy Law, $x > 0$, $x < 0$ or $x > 0$.

   Suppose $x > 0$.

   Then $|x| = x$, and $a|x| = ax$ by Substitution of Equals.

   Also, $ax > 0$ by EPI 10 and 1, so $|ax| = ax$.

   Hence, by Symmetry and Transitivity of Equals, $|ax| = a|x|$.

   Suppose $x = 0$.

   Then $|x| = |0| = 0$ and $a|x| = 0$ by EPI 1.

   Also, $ax = 0$ by EPI 1 and so $|ax| = 0$.

   Hence, by Symmetry and Transitivity of Equals, $|ax| = a|x|$.

   Suppoe $x < 0$.

   Then $|x| = -x$ and $a|x| = a(-x)$ by Subsitution of Equals.

   Also, $ax < a \cdot 0$ by EPI 10, so $ax < 0$ by EPI 1.

   So, $|ax| = -(ax)$.

   By EPI 3, $-(ax) = (-1)ax$.

   Also, $a(-x) = a(-1)x = (-1)ax$ by Commutativity and EPI 3.

   Thus, by Symmetry and Transitivity of Equals, $|ax| = a|x|$.

   Therefore, in all cases, $|ax| = a|x|$.■

2. **Theorem:** Let A, B and C be sets.
   Suppose $A \cup C \subseteq B \cup C$.
   Then $A \setminus C \subseteq B$.
   Proof: Let A, B and C be sets.
   Suppose $A \cup C \subseteq B \cup C$.
   Suppose $x \in A \setminus C$.
   Then $x \in A$ and $x \notin C$.
   Then $x \in A \cup C$, and so $x \in B \cup C$.
   So $x \in B$ or $x \in C$.
   Since $x \notin C$, $x \in B$.
   Thus, $x \in A \setminus C$ implies $x \in B$,
   $A \setminus C \subseteq B$. ■

3. **Theorem:** Let A, B and C be sets.

Suppose $A \subseteq B \cup C$ and $B \subseteq A \cup C$.

Then $A \cup B \cup C = C \cup (A \cap B)$.

Proof:

Let A, B and C be sets.

Suppose $A \subseteq B \cup C$ and $B \subseteq A \cup C$.

Suppose $x \in A \cup B \cup C$.

Then $x \in A$ or $x \in B$ or $x \in C$.

Suppose $x \in C$.

  Then $x \in C \cup (A \cap B)$.

Suppose $x \in A$.

  Then $x \in B \cup C$ (since $A \subseteq B \cup C$).

  So $x \in B$ or $x \in C$.

  Suppose $x \in C$.

    Then $x \in C \cup (A \cap B)$.

  Suppose $x \in B$.

    Then $x \in A \cap B$, so $x \in C \cup (A \cap B)$.

Suppose $x \in B$.

  Then $x \in A \cup C$ (since $B \subseteq A \cup C$).

  So $x \in A$ or $x \in C$.

  Suppose $x \in C$.

    Then $x \in C \cup (A \cap B)$.

  Suppose $x \in A$.

    Then $x \in A \cap B$, so $x \in C \cup (A \cap B)$.

Hence, $x \in A \cup B \cup C$ implies $x \in C \cup (A \cap B)$, so $A \cup B \cup C \subseteq C \cup (A \cap B)$.

Suppose $x \in C \cup (A \cap B)$.

Then $x \in C$ or $x \in A \cap B$.

Suppose $x \in C$.

  Then $x \in A \cup B \cup C$.

Suppose $x \in A \cap B$.

  Then $x \in A$, so $x \in A \cup B \cup C$.

Hence, $x \in C \cup (A \cap B)$ implies $x \in A \cup B \cup C$, so $C \cup (A \cap B) \subseteq A \cup B \cup C$.

Thus, $A \cup B \cup C = C \cup (A \cap B)$. ■

4. **Theorem:** Let a and b be integers. Then $a^2 = b^2$ if and only if $a = b$ or $a = -b$.

Proof: Let a and b be integers.

Then $(a-b)(a+b) = (a-b)a + (a-b)b$ by the Distributive axiom.

Also, $(a-b)a + (a-b)b = a^2 - ba + ab + (-b)b$ by the Distributive axiom.

Then, by Commutativity and EPI 3, $a^2 - ba + ab + (-b)(b) = a^2 - ba + ba - b^2$.

By the Additive Inverses axiom, $a^2 - ba + ba - b^2 = a^2 - b^2$, and so by Transitivity of Equals, $(a-b)(a+b) = a^2 - b^2$.

Suppose $a^2 = b^2$.

Then $a^2 - b^2 = 0$ (by the Additive Inverses axiom), and so $(a-b)(a+b) = 0$ by Transitivity of Equals.

By EPI 6, $a - b = 0$ or $a + b = 0$.

Hence, $a = b$ or $a = -b$ by the Additive Inverses axiom and Substitution of Equals.

So $a^2 = b^2$ implies $a = b$ or $a = -b$.

Suppose $a = b$.

Then $a^2 = ba$ by Substitution of Equals.

Also, $ab = b^2$ by Substitution of Equals.

Hence, $a^2 = b^2$ by Commutativity and Transitivity of Equals.

Suppose $a = -b$.

Then $a^2 = -ba$ by Substitution of Equals.

Also, $a(-b) = (-b)^2$ by Substitution of equals.

We have $(-b)^2 = (-b)(-b) = b^2$ by EPI 5.

Also, $a(-b) = a(-1)b = (-1)ab = (-1)ba = -ab$ by Commutativity and EPI 3.

Hence, by Transitivity of Equals, $a^2 = b^2$.

Thus, $a^2 = b^2$ iff $a = b$ or $a = -b$. ■