Math 300 C - Spring 2016
Midterm Exam Number Two
May 18, 2016
Solutions

1. *Let $A = \mathbb{R} \times \mathbb{R} \setminus \{(0,0)\}$.*

   Thus, A is the xy-plane without the origin.

   Define a relation R on A by

   $((x_1, y_1), (x_2, y_2)) \in R \Leftrightarrow (x_1, y_1)$ and (x_2, y_2) lie on a line which passes through the origin.

   Prove that R is transitive.

   Suppose $P = (x_1, y_1) \in A$ and $Q = (x_2, y_2) \in A$ and $S = (x_3, y_3) \in A$ and $(P, Q) \in R$ and $(Q, S) \in R$.

   Suppose $x_1 = 0$.

   Then P lies on the vertical line $x = 0$ through the origin and no other line through the origin.

   Hence, Q lies on $x = 0$, and hence S lies on $x = 0$.

   Thus, P and S lie on a line through the origin, and so $(P, S) \in R$.

   Suppose $x_1 \neq 0$.

   Then P and Q lie on a line $y = mx$ where $m \in \mathbb{R}$, and Q and R lie on a line $y = nx$ where $n \in \mathbb{R}$.

   Then $y_2 = mx_2 = nx_2$ so $(m - n)x_2 = 0$.

   If $x_2 = 0$, then $y_2 = mx_2 = 0$, so $Q = (0, 0) \notin A$, a contradiction since $Q \in A$.

   Hence $x_2 \neq 0$, so $m - n = 0$, i.e., $m = n$.

   Thus, P and Q lie on the line $y = mx$ and Q and S lie on the line $y = mx$, so P and S both lie on a line through the origin.

   Hence, $(P, S) \in R$ and thus R is transitive.

2. *Let A, B, and C be sets.*

   Let $f : A \to C$ and $g : B \to C$.

   Prove that $f \cup g : A \cup B \to C$ iff $f(x) = g(x)$ for all $x \in A \cap B$.

   Suppose $f \cup g : A \cup B \to C$.

   Suppose $\exists a \in A \cap B$ such that $f(x) \neq g(x)$.

   Then, since $a \in A$, $(a, f(a)) \in f$ and, since $a \in B$, $(a, g(a)) \in g$.

   Hence, $(a, f(a)) \in f \cup g$ and $(a, g(a)) \in f \cup g$.

   But $f(a) \neq g(a)$ and hence $\nexists ! y \in C$ such that $(a, y) \in f \cup g$, so $f \cup g$ is not a function.

   This is a contradiction to our assumption that $f \cup g : A \cup B \to C$.

   Hence, $f(x) = g(x)$, and so $f(x) = g(x) \forall x \in A \cap B$.

   Suppose $f(x) = g(x)$ for all $x \in A \cap B$.

   Suppose $x \in A \cup B$. So $x \in A$ or $x \in B$.

   Suppose $x \in A$.

   Then $(x, f(x)) \in f$, and so $(x, f(x) \in f \cup g$.

   Suppose $x \in B$.

   Then $(x, g(x)) \in g$ and so $(x, g(x) \in f \cup g$.

   Thus, $\exists y \in C$ such that $(x, c) \in f \cup g$.

   Suppose $(a, y_1) \in f \cup g$ and $(a, y_2) \in f \cup g$.

   Suppose $(a, y_1) \in f$ and $(a, y_2) \in f$. Then $f(a) = y_1$ and $f(a) = y_2$, so $y_1 = y_2$.

   Suppose $(a, y_1) \in g$ and $(a, y_2) \in g$. Then $g(a) = y1$ and $g(a) = y_2$, so $y_1 = y_2$.

   Suppose $(a, y_1) \in f$ and $(a, y_2) \in g$. Then $a \in A \cap B$, so $f(a) = y_1$, $g(a) = y_2$, and $f(a) = g(a)$, so $y_1 = y_2$.

   Suppose $(a, y_1) \in g$ and $(a, y_2) \in f$. Then $a \in A \cap B$, so $f(a) = y_2$, $g(a) = y_1$, and $f(a) = g(a)$, so $y_1 = y_2$.

   Hence, $y_1 = y_2$, and so, for each $a \in A \cup B$, there is a unique $y \in C$ such that $(a, y) \in f \cup g$.

   Thus, $f \cup g : A \to C$. ■

3. *Prove that* $18 \mid 49^n + 6n - 1$ *for all integers,* $n \geq 1$.

For integers $n \geq 1$, let $P(n)$ be the statement " $18 \mid 49^n + 6n - 1$ ".

Base case:

Let $n = 1$.

Then $49^n + 6n - 1 = 49 + 6 - 1 = 54 = (3)(18)$, so $18 \mid 49^n + 6n - 1$, and so $P(1)$ is true.

Induction step:

Suppose $P(n)$ is true for some $n = k \geq 1$.

So $P(k)$ is true.

Then $18 \mid 49^k + 6k - 1$, so $49^k + 6k - 1 = 18m$ for some integer m.

Then:

$$\begin{aligned} 49^{k+1} + 6(k+1) - 1 &= 49(49^k + 6k - 1) - 49(6k) - 49 + 6(k+1) - 1 \\ &= 49(49^k + 6k - 1) - 48(6k) + 54 \\ &= 49(18m) - 18(16k) + (3)(18) \\ &= 18(49m - 16k + 3). \end{aligned}$$

Since m and k are integers, $49m - 16k + 3$ is an integer, and so $18 \mid 49^{k+1} + 6(k+1) - 1$.

Thus $P(k+1)$ is true, and so $P(k)$ implies $P(k+1)$.

Hence, since $P(1)$ is true, by induction $P(n)$ is true for all integers $n \geq 1$. ■

4. *Let $\mathcal{F}$ and $\mathcal{G}$ be families (i.e., sets of sets).*

(a) Prove that

$$\cup\mathcal{F} \setminus \cup\mathcal{G} \subseteq \cup(\mathcal{F} \setminus \mathcal{G}).$$

Suppose $x \in \cup\mathcal{F} \setminus \cup\mathcal{G}$.
Then $x \in \cup\mathcal{F}$ and $x \notin \cup\mathcal{G}$.
Hence $\exists A \in \mathcal{F}$ such that $x \in A$.
Since $x \notin \cup\mathcal{G}$, $A \notin \mathcal{G}$.
Hence, $A \in \mathcal{F} \setminus \mathcal{G}$, so $x \in \cup(\mathcal{F} \setminus \mathcal{G})$.

Thus, $x \in \cup\mathcal{F} \setminus \cup\mathcal{G}$ implies $x \in \cup(\mathcal{F} \setminus \mathcal{G})$, and so $\cup\mathcal{F} \setminus \cup\mathcal{G} \subseteq x \in \cup(\mathcal{F} \setminus \mathcal{G})$. ■

(b) *Give an example of non-empty families $\mathcal{F}$ and $\mathcal{G}$ such that*

$$\cup\mathcal{F} \setminus \cup\mathcal{G} \neq \cup(\mathcal{F} \setminus \mathcal{G}).$$

Prove that your example is valid.
Let $\mathcal{F} = \{\{1,2\},\{2\}\}$ and $\mathcal{G} = \{\{2\}\}$.
Then

$$\cup\mathcal{F} \setminus \cup\mathcal{G} = \{1,2\} \setminus \{2\} = \{1\}$$

and

$$\cup(\mathcal{F} \setminus \mathcal{G}) = \cup\{\{1,2\}\} = \{1,2\}.$$

Since $\{1\} \neq \{1,2\}$, $\cup\mathcal{F} \setminus \cup\mathcal{G} \neq \cup(\mathcal{F} \setminus \mathcal{G}$ ■