Math 300 C - Spring 2022
Midterm Exam Number Two
May 18, 2022

Name: ______________________________ Student ID no. : ________________

Signature: ____

- You have 50 minutes to complete the exam.
- When time is called, you must stop writing immediately.

1. Prove the following theorem.

   Theorem: Let A, B, and C be sets. Suppose $A \cup C \subseteq B \cup C$. Then $A \setminus C \subseteq B$.

2. Prove the following theorem.

   Theorem: Suppose $\mathcal{A}, \mathcal{B}$, and $\mathcal{C}$ are families of sets, with $\mathcal{A} \neq \varnothing$, $\mathcal{B} \neq \varnothing$, and $\mathcal{C} \neq \varnothing$.
   Suppose that for every $S \in \mathcal{A}$ and $T \in \mathcal{B}$, $S \cup T \in \mathcal{C}$.
   Then $\bigcap \mathcal{C} \subseteq (\bigcap \mathcal{A}) \cup (\bigcap \mathcal{B})$.

3. Use induction to prove the following theorem.

   Theorem: For all integers $n \geq 1$,

$$\sum_{j=1}^{n} \frac{1}{(2j-1)(2j+1)} = \frac{n}{2n+1}.$$

4. Define the relation $R \subseteq \mathbb{Z} \times \mathbb{Z}$ by

$$(x, y) \in R \text{ if and only if } 3 \mid (x - y).$$

Is R reflexive?

Is R symmetric?

Is R transitive?

Is R an equivalence relation?

(Note you have **four** questions to answer here.)

Elementary Properties of the Integers (EPIs)

Suppose a, b, c, and d are integers.

1. **Closure:** $a + b$ and ab are integers.
2. **Substitution of Equals:** If $a = b$, then $a + c = b + c$ and $ac = bc$.
3. **Commutativity:** $a + b = b + a$ and $ab = ba$.
4. **Associativity:** $(a + b) + c = a + (b + c)$ and $(ab)c = a(bc)$.
5. **The Distributive Law:** $a(b + c) = ab + ac$
6. **Identities:** $a + 0 = 0 + a = a$ and $a \cdot 1 = 1 \cdot a = a$.

   0 is called the additive identity.

   1 is called the multiplicative identity.
7. **Additive Inverses:** There exists an integer $-a$ such that $a + (-a) = (-a) + a = 0$.
8. **Trichotomy:** Exactly one of the following is true:
   $a > 0$, $-a > 0$, or $a = 0$.
9. **The Well-Ordering Principle:** Every non-empty set of positive integers contains a smallest element.
10. $a \cdot 0 = 0$
11. If $a + c = b + c$, then $a = b$.
12. $-a = (-1) \cdot a$
13. $(-a) \cdot b = -(ab)$
14. $(-a) \cdot (-b) = ab$
15. If $ab = 0$, then $a = 0$ or $b = 0$.
16. If $a \leq b$ and $b \leq a$, then $a = b$.
17. If $a < b$ and $b < c$, then $a < c$.
18. If $a < b$, then $a + c < b + c$.
19. If $a < b$ and $0 < c$, then $ac < bc$.
20. If $a < b$ and $c < 0$, then $bc < ac$.
21. If $a < b$ and $c < d$, then $a + c < b + d$.
22. If $0 \leq a < b$ and $0 \leq c < d$, then $ac < bd$.
23. If $a < b$, then $-b < -a$.
24. $0 \leq a^2$, where $a^2 = a \cdot a$.
25. If $ab = 1$, then either $a = b = 1$ or $a = b = -1$.

Sets

$A \subseteq B$ iff $x \in A$ implies $x \in B$

$A = B$ iff $A \subseteq B$ and $B \subseteq A$

$x \in A \cup B$ iff $x \in A$ or $x \in B$

$x \in A \cap B$ iff $x \in A$ and $x \in B$

$x \in A \setminus B$ iff $x \in A$ and $x \notin B$

NOTE: Properties 17-23 hold if each $<$ is replaced with $\leq$.

One theorem for reference:
Theorem DAS (Divisors are Smaller):
Let a and b be positive integers. Then $a|b$ implies $a \leq b$.