Math 300 D - Winter 2014
Midterm Exam
January 29, 2014
Answers

1. *Let A,B,and C be sets. Verify the following identities by showing that that the statement $x \in$ "the left hand set" is equivalent to the statement $x \in$ "the right hand set". Give a justification for each step (e.g., definition of union, DeMorgan's law, distributive law, etc.)*

   (a) $A \cap (B \cup (A \cap B)) = A \cap B$

$$\begin{aligned}
& x \in A \cap (B \cup (A \cap B)) & \\
\Leftrightarrow\ & x \in A \wedge (x \in B \vee x \in A \cap B) & \text{(def. of } \cap, \cup) \\
\Leftrightarrow\ & x \in A \wedge (x \in B \vee (x \in A \wedge x \in B)) & \text{(def. of } \cap) \\
\Leftrightarrow\ & x \in A \wedge x \in B & \text{(absorption law)} \\
\Leftrightarrow\ & x \in (A \cap B) & \text{(def. of } \cap)
\end{aligned}$$

   (b) $(A \cap B) \setminus (B \cap C) = (A \cap B) \setminus C$

$$\begin{aligned}
& x \in (A \cap B) \setminus (B \cap C) & \\
\Leftrightarrow\ & x \in A \cap B \wedge x \notin B \cap C & \text{(def. of } \setminus) \\
\Leftrightarrow\ & x \in A \wedge x \in B \wedge \neg(x \in B \wedge x \in C) & \text{(def. of } \cap, \notin) \\
\Leftrightarrow\ & x \in A \wedge x \in B \wedge (x \notin B \vee x \notin C) & \text{(DeMorgan's law)} \\
\Leftrightarrow\ & x \in A \wedge (x \in B \wedge (x \notin B \vee x \notin C)) & \text{(associative laws)} \\
\Leftrightarrow\ & x \in A \wedge ((x \in B \wedge x \notin B) \vee (x \in B \wedge x \notin C)) & \text{(distibution laws)} \\
\Leftrightarrow\ & x \in A \wedge (x \in B \wedge x \notin C) & \text{(tautology laws)} \\
\Leftrightarrow\ & (x \in A \wedge x \in B) \wedge x \notin C & \text{(associative laws)} \\
\Leftrightarrow\ & x \in (A \cap B) \setminus C & \text{(def. of } \cap, \setminus)
\end{aligned}$$

2. *Let A,B,C and D be sets. Verify the following identity by showing that that the statement $x \in$ "the left hand set" is equivalent to the statement $x \in$ "the right hand set". Give a justification for each step (e.g., definition of union, DeMorgan's law, distributive law, etc.)*

$$((A \cup B) \setminus (B \setminus A)) \setminus A = \varnothing$$

$$\begin{aligned}
& x \in ((A \cup B) \setminus (B \setminus A)) \setminus A \\
\Leftrightarrow\ & (x \in A \cup B \wedge x \notin B \setminus A) \wedge x \notin A && \text{(def. of } \setminus) \\
\Leftrightarrow\ & (x \in A \vee x \in B) \wedge \neg(x \in B \wedge x \notin A) \wedge x \notin A && \text{(def. of } \cap, \setminus) \\
\Leftrightarrow\ & (x \in A \vee x \in B) \wedge (x \notin B \vee x \in A) \wedge x \notin A && \text{(DeMorgan's law)} \\
\Leftrightarrow\ & (x \in A \vee (x \in B \wedge x \notin B)) \wedge x \notin A && \text{(distrubtive laws)} \\
\Leftrightarrow\ & x \in A \wedge x \notin A && \text{(contradiction laws)} \\
\Leftrightarrow\ & x \in \varnothing
\end{aligned}$$

In the last step, we use the fact that all contradictions are equivalent.

3. *Simplify the following expressions as much as possible. You should show a sequence of equivalent expressions connecting the original expression with your final one, and give a justification for each step (e.g., DeMorgan's law, distributive law, etc.)*

(a) $P \vee (\neg P \wedge (P \vee Q))$

$$\begin{aligned}
& P \vee (\neg P \wedge (P \vee Q)) \\
\Leftrightarrow\ & (P \vee \neg P) \wedge (P \vee (P \vee Q)) && \text{(distributive laws)} \\
\Leftrightarrow\ & P \vee (P \vee Q) && \text{(tautology laws)} \\
\Leftrightarrow\ & P \vee Q && \text{(idempotent laws)}
\end{aligned}$$

(b) $(\neg P \wedge (Q \vee P)) \vee (P \vee \neg Q)$

$$\begin{aligned}
& (\neg P \wedge (Q \vee P)) \vee (P \vee \neg Q) \\
\Leftrightarrow\ & ((\neg P \wedge Q) \vee (\neg P \wedge P)) \vee (P \vee \neg Q) && \text{(distributive law)} \\
\Leftrightarrow\ & (\neg P \wedge Q) \vee (P \vee \neg Q) && \text{(contradiction law)} \\
\Leftrightarrow\ & ((\neg P \wedge Q) \vee P) \vee \neg Q && \text{(associative law)} \\
\Leftrightarrow\ & ((\neg P \vee P) \wedge (Q \vee P)) \vee \neg Q && \text{(distributive law)} \\
\Leftrightarrow\ & (Q \vee P) \vee \neg Q && \text{(tautology law)} \\
\Leftrightarrow\ & \text{tautology} && \text{(associative/commutative/tautology laws)}
\end{aligned}$$

4. *Simplify the following expressions as much as possible. You should show a sequence of equivalent expressions connecting the original expression with your final one, and give a justification for each step (e.g., DeMorgan's law, distributive law, etc.)*

(a) $P \leftrightarrow (Q \to (P \vee R))$

$P \leftrightarrow (Q \to (P \vee R))$	
$\Leftrightarrow (P \to (Q \to (P \vee R))) \wedge ((Q \to (P \vee R)) \to P)$	(def. of $\leftrightarrow$)
$\Leftrightarrow (\neg P \vee (\neg Q \vee P \vee R)) \wedge ((\neg Q \vee (P \vee R)) \vee P)$	(conditional laws)
$\Leftrightarrow (Q \wedge \neg(P \vee R)) \vee P)$	(tautology and DeMorgan's laws)
$\Leftrightarrow (Q \wedge \neg R \wedge \neg P) \vee P$	(DeMorgan's and commutative laws)
$\Leftrightarrow (P \vee (Q \wedge \neg R)) \wedge (P \vee \neg P)$	(distributive and commutative laws)
$\Leftrightarrow P \vee (Q \wedge \neg R)$	(tautology laws)

(b) $(P \wedge \neg Q) \to (P \to Q)$

$(P \wedge \neg Q) \to (P \to Q)$	
$\Leftrightarrow \neg(P \wedge \neg Q) \vee (\neg P \vee Q)$	(conditional laws)
$\Leftrightarrow \neg P \vee Q \vee \neg P \vee Q$	(DeMorgan's laws)
$\Leftrightarrow \neg P \vee Q$	(idempotent and commutative laws)

5. *Express the following using only the connectives $\neg$ and $\wedge$:*

$$(P \leftrightarrow Q) \vee R$$

$(P \leftrightarrow Q) \vee R$	
$\Leftrightarrow ((P \to Q) \wedge (Q \to P)) \vee R$	(def. of $\leftrightarrow$)
$\Leftrightarrow ((\neg P \vee Q) \wedge (\neg Q \vee P)) \vee R$	(conditional laws)
$\Leftrightarrow \neg(\neg(\neg(P \wedge \neg Q) \wedge \neg(Q \wedge \neg P)) \wedge \neg R)$	(DeMorgan's laws)

6. *For a set A, let $\mathcal{P}(A)$ be the power set of A. Write out the set given by the expression*

$$\mathcal{P}(\{1,2,3\}) \cap \mathcal{P}(\{2,3,4\}).$$

$$\mathcal{P}(\{1,2,3\}) \cap \mathcal{P}(\{2,3,4\}) = \{\varnothing, \{2\}, \{3\}, \{2,3\}\}.$$