Math 300 B - Winter 2015
Midterm Exam Number Two
February 25, 2015

Name: ______________________ Student ID no. : ______________

Signature: ____

1	10	
2	10	
3	10	
4	10	
5	10	
Total	50	

- Complete all 5 questions.
- You have 50 minutes to complete the exam.

1. Let A, B and C be sets. Prove that $(A \cup B) \cap C \subseteq A \cup (B \cap C)$.

2. Let A and B be sets. Prove that $A \cap B = \varnothing$ if and only if $\mathcal{P}(A) \cap \mathcal{P}(B) = \{\varnothing\}$.

3. Prove that, for all integers n, 3 does not divide $n^2 - 5$.

4. Let A and B be sets. Prove that $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$.

5. Define a relation R on $\mathbb{Z}$ by

$$(x, y) \in R \Leftrightarrow 4 \mid x^2 - y^2.$$

Is R an equivalence relation? Prove your answer.

Sets

Elementary Properties of Real Numbers

$$A = B \Leftrightarrow ((x \in A) \Leftrightarrow (x \in B))$$
$$A \subseteq B \Leftrightarrow ((x \in A) \rightarrow (x \in B))$$
$$x \in A \cup B \Leftrightarrow ((x \in A) \vee (x \in B))$$
$$x \in A \cap B \Leftrightarrow ((x \in A) \wedge (x \in B))$$
$$x \in A \setminus B \Leftrightarrow (x \in A) \wedge (x \notin B)$$

Axioms

Suppose x, y, and z are real numbers. We will take as fact each of the following.

1. $x + y$ and xy are real numbers. ($\mathbb{R}$ is *closed* under addition and multiplication.)
2. If $x = y$, then $x + z = y + z$ and $xz = yz$. (This is sometimes called *substitution of equals*.)
3. $x + y = y + x$ and $xy = yx$ (addition and multiplication are *commutative* in $\mathbb{R}$)
4. $(x + y) + z = x + (y + z)$ and $(xy)z = x(yz)$ (addition and multiplication are *associative* in $\mathbb{R}$)
5. $x(y + z) = xy + xz$ (This is the *Distributive Law*.)
6. $x + 0 = 0 + x = x$ and $x \cdot 1 = 1 \cdot x = x$ (0 is the *additive identity*; 1 is the *multiplicative identity*.)
7. There exists a real number $-x$ such that $x + (-x) = (-x) + x = 0$. (That is, every real number has an *additive inverse* in $\mathbb{R}$.)
8. If $x \neq 0$, then there exists a real number x^{-1} such that $x \cdot x^{-1} = x^{-1} \cdot x = 1$. (That is, every non-zero real number has a *multiplicative inverse* in $\mathbb{R}$.)
9. If $x > 0$ and $y > 0$, then $x + y > 0$ and $xy > 0$.
10. Either $x > 0$, $-x > 0$, or $x = 0$.
11. If x and y are integers, then $-x$, $x + y$, and xy are integers. (The additive inverse of an integer is an integer and $\mathbb{Z}$ is closed under addition and multiplication.)

NOTE: It is not hard to prove that $\mathbb{Q}$, the set of rational numbers is closed under addition and multiplication and that every non-zero rational number has a multiplicative inverse in $\mathbb{Q}$.

The following properties of real numbers that allow us to do algebra follow from the axioms on the front page.

If x, y, z, u, and v are real numbers, then:

1. $x \cdot 0 = 0$
2. If $x + z = y + z$, then $x = y$.
3. If $x \cdot z = y \cdot z$ and $z \neq 0$, then $x = y$.
4. $-x = (-1) \cdot x$
5. $(-x) \cdot y = -(x \cdot y)$
6. $(-x) \cdot (-y) = x \cdot y$
7. If $x \cdot y = 0$, then $x = 0$ or $y = 0$.
8. If $x \leq y$ and $y \leq x$, then $x = y$.
9. If $x \leq y$ and $y \leq z$, then $x \leq z$.
10. At least one of the following is true: $x \leq y$ or $y \leq x$.
11. If $x \leq y$, then $x + z \leq y + z$.
12. If $x \leq y$ and $0 \leq z$, then $xz \leq yz$.
13. If $x \leq y$ and $z \leq 0$, then $yz \leq xz$.
14. If $x \leq y$ and $u \leq v$, then $x + u \leq y + v$.
15. If $0 \leq x \leq y$ and $0 \leq u \leq v$, then $xu \leq yv$.
16. If $x \leq y$, then $-y \leq -x$.
17. $0 \leq x^2$
18. $0 < 1$
19. If $0 < x$, then $0 < x^{-1}$.
20. If $0 < x < y$, then $0 < y^{-1} < x^{-1}$.

And here are a couple of properties of integers.

21. Every integer is either even or odd, never both.
22. The only integers that divide 1 are -1 and 1.