Math 300 B - Winter 2015
Midterm Exam Number Two
February 25, 2015
Answers

1. *Let A, B and C be sets. Prove that $(A \cup B) \cap C \subseteq A \cup (B \cap C)$.*

   Let A, B, and C be sets.

   Suppose $x \in (A \cup B) \cap C$.

   Then $x \in (A \cup B)$ and $x \in C$.

   So, $x \in A$ or $x \in B$.

   Suppose $x \in A$.

   Then $x \in A \cup (B \cap C)$.

   Suppose $x \in B$.

   Then $x \in B \cap C$, so $x \in A \cup (B \cap C)$.

   Therefore $x \in A \cup (B \cap C)$.

   Thus, $x \in (A \cup B) \cap C$ implies $x \in A \cup (B \cap C)$.

   Hence, $(A \cup B) \cap C \subseteq A \cup (B \cap C)$. ■

2. *Let A and B be sets. Prove that $A \cap B = \varnothing$ if and only if $\mathcal{P}(A) \cap \mathcal{P}(B) = \{\varnothing\}$.*

   Let A and B be sets.

   Suppose $A \cap B \neq \varnothing$.

   Then there exists x such that $x \in A$ and $x \in B$.

   Let $S = \{x\}$, the set containing x as its only element.

   Since $x \in A$, $S \subseteq A$, so $S \in \mathcal{P}(A)$.

   Since $x \in B$, $S \subseteq B$, so $S \in \mathcal{P}(B)$.

   Thus, $S \in \mathcal{P}(A) \cap \mathcal{P}(B)$, and $S \neq \varnothing$, so $\mathcal{P}(A) \cap \mathcal{P}(B) \neq \{\varnothing\}$.

   Suppose $\mathcal{P}(A) \cap \mathcal{P}(B) \neq \{\varnothing\}$.

   Since $\varnothing \in \mathcal{P}(A) \cap \mathcal{P}(B)$, we can conclude that there is a set $S \in \mathcal{P}(A) \cap \mathcal{P}(B)$ and $S \neq \varnothing$.

   Hence, there is an element $x \in S$.

   Since $S \in \mathcal{P}(A)$, $S \subseteq A$, and so $x \in A$.

   Since $S \in \mathcal{P}(B)$, $S \subseteq B$, and so $x \in B$.

   Thus, $x \in A \cup B$, so $A \cup B \neq \varnothing$.

   Hence, $A \cap B = \varnothing$ if and only if $\mathcal{P}(A) \cap \mathcal{P}(B) = \{\varnothing\}$.■

3. *Prove that, for all integers n, 3 does not divide $n^2 - 5$.*

   Let n be an integer.

   By Euclid's theorem, $n = 3k + r$ for some integers k and r, and $0 \leq r < 3$.

   That is, $r = 0, 1$, or 2.

   Then $n^2-5 = (3k+r)^2-5 = 9k^2+6kr+r^2-5 = 9k^2+6kr+r^2-6+1 = 3(3k^2+2kr-2)+r^2+1$.

   Let t be the remainder when $n^2 - 5$ is divided by 3.

   Then t is equal to the remainder then $r^2 + 1$ is divided by 3.

   If $r = 0$, then $r^2 + 1 = 1 = 0 \cdot 3 + 1$, so $t = 1$.

   If $r = 1$, then $r^2 + 1 = 2 = 0 \cdot 3 + 2$, so $t = 2$.

   If $r = 2$, then $r^2 + 1 = 5 = 1 \cdot 3 + 2$, so $t = 2$.

   Hence the remainder when $n^2 - 5$ is divided by 3 is 1 or 2 and not zero.

   Thus 3 does not divide $n^2 - 5$. ■

4. *Let A and B be sets. Prove that $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$.*

   Let A and B be sets.

   Suppose $S \in \mathcal{P}(A \cap B)$.

   Then $S \subseteq A \cap B$.

   Suppose $x \in S$.

   Then $x \in A \cap B$, so $x \in A$ and $x \in B$.

   Thus $x \in S$ implies $x \in A$, so $S \subseteq A$.

   Also, $x \in S$ implies $x \in B$, so $S \subseteq B$.

   Thus $S \in \mathcal{P}(A)$ and $S \in \mathcal{P}(B)$, so $S \in \mathcal{P}(A) \cap \mathcal{P}(B)$.

   Thus $S \in \mathcal{P}(A \cap B)$ implies $S \in \mathcal{P}(A) \cap \mathcal{P}(B)$, so $\mathcal{P}(A \cap B) \subseteq \mathcal{P}(A) \cap \mathcal{P}(B)$.

   Suppose $S \in \mathcal{P}(A) \cap \mathcal{P}(B)$.

   Then $S \in \mathcal{P}(A)$ and $S \in \mathcal{P}(B)$.

   So $S \subset A$ and $S \subset B$.

   Suppose $x \in S$.

   Then $x \in A$ and $x \in B$, so $x \in A \cap B$.

   Thus, $x \in S$ implies $x \in A \cap B$, so $S \subseteq A \cap B$.

   That is, $S \in \mathcal{P}(A \cap B)$.

   Thus, $S \in \mathcal{P}(A) \cap \mathcal{P}(B)$ implies $S \in \mathcal{P}(A \cap B)$, so

   $$\mathcal{P}(A) \cap \mathcal{P}(B) \subseteq \mathcal{P}(A \cap B).$$

   Since $\mathcal{P}(A) \cap \mathcal{P}(B) \subseteq \mathcal{P}(A \cap B)$ and $\mathcal{P}(A \cap B) \subseteq \mathcal{P}(A) \cap \mathcal{P}(B)$,

   $$\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B). \blacksquare$$

5. *Define a relation R on $\mathbb{Z}$ by*

$$(x, y) \in R \Leftrightarrow 4 \mid x^2 - y^2.$$

Is R an equivalence relation? Prove your answer.

Let $x \in \mathbb{Z}$.

Then $x^2 - x^2 = 0 = 4 \cdot 0$, so $4 \mid x^2 - x^2$.

Hence, $(x, x) \in R$, and so R is reflexive.

Suppose $(x, y) \in R$.

Then $4 \mid x^2 - y^2$.

That is, $x^2 - y^2 = 4k$ for some integer k.

Then $y^2 - x^2 = -4k = 4(-k)$ and since $-k \in \mathbb{Z}$, $4 \mid y^2 - x^2$.

Hence, $(y, x) \in R$, and so R is symmetric.

Suppose $(x, y) \in R$ and $(y, z) \in R$.

Then $4 \mid x^2 - y^2$, so $x^2 - y^2 = 4k$ for some integer k.

Also, $4 \mid y^2 - z^2$, so $y^2 - z^2 = 4m$ for some integer m.

Hence, $x^2 - y^2 + y^2 - z^2 = x^2 - z^2 = 4k + 4m = 4(k + m)$.

Since $k + m$ is an integer, $4 \mid x^2 - z^2$, and so $(x, z) \in R$.

Thus, R is transitive.

Since R is reflexive, symmetric and transitive, R is an equivalence relation. ■