Math 300 E - Winter 2019
Midterm Exam Number Two
February 27, 2019

Name: ______________________________ Student ID no. : ______________________

Signature: ____

1	10	
2	10	
3	10	
4	10	
5	10	
Total	50	

- You have 50 minutes to complete the exam.

1. Let A and B be sets. Prove that $\mathcal{P}(A \setminus B) \subseteq \mathcal{P}(A) \setminus \mathcal{P}(B)$.

2. Let $\mathcal{F}$ and $\mathcal{G}$ be non-empty families (i.e., sets of sets). Prove that $(\bigcap \mathcal{F}) \cap (\bigcap \mathcal{G}) = \bigcap (\mathcal{F} \cup \mathcal{G})$.

3. Let A, B and C be sets. Prove that $A \cup C \subseteq B \cup C$ iff $A \setminus C \subseteq B \setminus C$.

4. Use induction to prove that, for all integers $n \geq 0$,

$$\sum_{i=0}^{n} \frac{1}{3^i} = \frac{3}{2} - \frac{1}{2 \cdot 3^n}.$$

5. Let $R \subseteq \mathbb{R} \times \mathbb{R}$ be defined by

$$(x, y) \in R \text{ iff } x - y \in \mathbb{Z} \text{ or } x + y \in \mathbb{Z}.$$

Is R an equivalence relation? Support your answer with a proof.

Axioms of the Integers (AIs)

Suppose a, b, and c are integers.

- **Closure:**

  $a+b$ and ab are integers.

- **Substitution of Equals:**

  If $a=b$, then $a+c=b+c$ and $ac=bc$.

- **Commutativity:**

  $a+b=b+a$ and $ab=ba$.

- **Associativity:**

  $(a+b)+c=a+(b+c)$ and $(ab)c=a(bc)$.

- **The Distributive Law:**

  $a(b+c)=ab+ac$

- **Identities:**

  $a+0=0+a=a$ and $a\cdot 1=1\cdot a=a$

  0 is called the *additive identity*

  1 is called the *multiplicative identity*.

- **Additive Inverses:**

  There exists an integer $-a$ such that $a+(-a)=(-a)+a=0$.

- **Trichotomy:**

  Exactly one of the following is true: $a<0$, $-a<0$, or $a=0$.

Sets

$A\subseteq B$ iff $x\in A$ implies $x\in B$

$A=B$ iff $A\subseteq B$ and $B\subseteq A$

$x\in A\cup B$ iff $x\in A$ or $x\in B$

$x\in A\cap B$ iff $x\in A$ and $x\in B$

$x\in A\setminus B$ iff $x\in A$ and $x\notin B$

$\mathcal{P}(A)$ is the set of all subsets of a set A

Elementary Properties of the Integers (EPIs)

Suppose a, b, c, and d are integers.

1. $a\cdot 0=0$
2. If $a+c=b+c$, then $a=b$.
3. $-a=(-1)\cdot a$
4. $(-a)\cdot b=-(a\cdot b)$
5. $(-a)\cdot(-b)=a\cdot b$
6. If $a\cdot b=0$, then $a=0$ or $b=0$.
7. If $a\le b$ and $b\le a$, then $a=b$.
8. If $a<b$ and $b<c$, then $a<c$.
9. If $a<b$, then $a+c<b+c$.
10. If $a<b$ and $0<c$, then $ac<bc$.
11. If $a<b$ and $c<0$, then $bc<ac$.
12. If $a<b$ and $c<d$, then $a+c<b+d$.
13. If $0\le a<b$ and $0\le c<d$, then $ac<bd$.
14. If $a<b$, then $-b<-a$.
15. $0\le a^2$
16. If $ab=1$, then either $a=b=1$ or $a=b=-1$.

NOTE: Properties 8-14 hold if each $<$ is replaced with $\le$.

One theorem for reference:

Theorem DAS (Divisors are Smaller): Let a and b be positive integers. Then $a|b$ implies $a\le b$.