# Math 300 E - Winter 2019
# Midterm Exam Number Two
# February 27, 2019
# Answers

1. *Let $A$ and $B$ be sets. Prove that $\mathcal{P}(A \setminus B) \subseteq \mathcal{P}(A) \setminus \mathcal{P}(B)$.*

   **Proof:** Let $A$ and $B$ be sets.

   Suppose $S \in \mathcal{P}(A \setminus B)$.

   Then $S \subseteq A \setminus B$.

   Suppose $x \in S$.

   Then $x \in A$ and $x \notin B$.

   So $x \in S$ implies $x \in A$, and hence $S \subseteq A$.

   Thus, $S \in \mathcal{P}(A)$.

   Since $x \in S$ does not imply $x \in B$, we conclude that $S \notin B$, so $S \notin \mathcal{P}(B)$.

   Hence, $S \in \mathcal{P}(A) \setminus \mathcal{P}(B)$.

   Thus, $S \in \mathcal{P}(A \setminus B)$ implies $S \in \mathcal{P}(A) \setminus \mathcal{P}(B)$, and so $\mathcal{P}(A \setminus B) \subseteq \mathcal{P}(A) \setminus \mathcal{P}(B)$. ■

2. *Let $\mathcal{F}$ and $\mathcal{G}$ be non-empty families (i.e., sets of sets). Prove that $(\bigcap \mathcal{F}) \cap (\bigcap \mathcal{G}) = \bigcap (\mathcal{F} \cup \mathcal{G})$.*

**Proof:** Let $\mathcal{F}$ and $\mathcal{G}$ be non-empty families.

Suppose $x \in (\bigcap \mathcal{F}) \cap (\bigcap \mathcal{G})$.

Then $x \in \bigcap F$ and $x \in \bigcap G$.

Suppose $S \in \mathcal{F} \cup \mathcal{G}$ ( we are trying to show that $x$ is in $\bigcap (\mathcal{F} \cup \mathcal{G})$.)

Then $S \in \mathcal{F}$ or $S \in \mathcal{G}$.

If $S \in \mathcal{F}$, then $x \in S$, since $x \in \bigcap \mathcal{F}$.

If $S \in \mathcal{G}$, then $x \in S$, since $x \in \bigcap \mathcal{G}$.

Hence, $x \in S$.

So $S \in \mathcal{F} \cup \mathcal{G}$ implies $x \in S$, and so $x \in \bigcap (\mathcal{F} \cup \mathcal{G})$

Thus, $x \in (\bigcap \mathcal{F}) \cap (\bigcap \mathcal{G})$ implies $x \in \bigcap (\mathcal{F} \cup \mathcal{G})$, and hence $(\bigcap \mathcal{F}) \cap (\bigcap \mathcal{G}) \subseteq \bigcap (\mathcal{F} \cup \mathcal{G})$.

Now, suppose $x \in \bigcap (\mathcal{F} \cup \mathcal{G})$.

Suppose $S \in F$.

Then $S \in F \cup G$, so $x \in S$ since $x \in \bigcap (\mathcal{F} \cup \mathcal{G})$ .

Hence, $S \in F$ implies $x \in S$, and so $x \in \bigcap \mathcal{F}$.

Suppose $S \in G$.

Then $S \in F \cup G$, so $x \in S$ since $x \in \bigcap (\mathcal{F} \cup \mathcal{G})$ .

Hence, $S \in G$ implies $x \in S$, and so $x \in \bigcap \mathcal{G}$.

So, $x \in (\bigcap \mathcal{F}) \cap (\bigcap \mathcal{G})$.

Thus, $x \in \bigcap (\mathcal{F} \cup \mathcal{G})$ implies $x \in (\bigcap \mathcal{F}) \cap (\bigcap \mathcal{G})$, and hence $\bigcap (\mathcal{F} \cup \mathcal{G}) \subseteq (\bigcap \mathcal{F}) \cap (\bigcap \mathcal{G})$.

Therefore, $(\bigcap \mathcal{F}) \cap (\bigcap \mathcal{G}) = \bigcap (\mathcal{F} \cup \mathcal{G})$.■

3. *Let $A$, $B$ and $C$ be sets. Prove that $A \cup C \subseteq B \cup C$ iff $A \setminus C \subseteq B \setminus C$.*

**Proof:** Let $A$, $B$, and $C$ be sets.

Suppose $A \cup C \subseteq B \cup C$.

Suppose $x \in A \setminus C$.

Then $x \in A$ and $x \notin C$.

Then $x \in A \cup C$, and so $x \in B \cup C$.

That is, $x \in B$ or $x \in C$.

But, $x \notin C$, so $x \in B$.

Hence, $x \in B \setminus C$.

Thus, $x \in A \setminus C$ implies $x \in B \setminus C$, so $A \setminus C \subseteq B \setminus C$.

Thus, $A \cup C \subseteq B \cup C$ implies $A \setminus C \subseteq B \setminus C$.

Now, suppose $A \setminus C \subseteq B \setminus C$.

Suppose $x \in A \cup C$.

Then $x \in A$ or $x \in C$.

If $x \in C$, then $x \in B \cup C$.

Suppose $x \notin C$.

Then $x \in A$, so $x \in A \setminus C$.

Hence, $x \in B \setminus C$, and so $x \in B$, and thus $x \in B \cup C$.

Thus, $x \in A \cup C$ implies $x \in B \cup C$, so $A \cup C \subseteq B \cup C$.

Thus, $A \setminus C \subseteq B \setminus C$ implies $A \cup C \subseteq B \cup C$.

Hence, $A \cup C \subseteq B \cup C$ iff $A \setminus C \subseteq B \setminus C$. ■

4. *Use induction to prove that, for all integers $n \geq 0$,*

$$\sum_{i=0}^{n} \frac{1}{3^i} = \frac{3}{2} - \frac{1}{2 \cdot 3^n}.$$

**Proof:** For all integers $n \geq 0$, let $P(n)$ be the statement "$\sum_{i=0}^{n} \frac{1}{3^i} = \frac{3}{2} - \frac{1}{2 \cdot 3^n}$".

We proceed by induction.

**Base Case:** Let $n = 0$.

Then

$$\sum_{i=0}^{n} \frac{1}{3^i} = \sum_{i=0}^{0} \frac{1}{3^i} = \frac{1}{3^0} = 1 = \frac{3}{2} - \frac{1}{2} = \frac{3}{2} - \frac{1}{2 \cdot 3^0} = \frac{3}{2} - \frac{1}{2 \cdot 3^n}.$$

So $P(0)$ is true.

**Induction Step:** Suppose $P(k)$ is true for some $k \in \mathbb{Z}_{\geq 0}$.

Then

$$\begin{aligned}\sum_{i=0}^{k+1} \frac{1}{3^i} &= \frac{1}{3^{k+1}} + \sum_{i=0}^{k} \frac{1}{3^i} \\ &= \frac{1}{3^{k+1}} + \frac{3}{2} - \frac{1}{2 \cdot 3^k} \\ &= \frac{3}{2} + \frac{2}{2 \cdot 3^{k+1}} - \frac{3}{2 \cdot 3^{k+1}} \\ &= \frac{3}{2} - \frac{1}{2 \cdot 3^{k+1}}.\end{aligned}$$

Hence, $P(k+1)$ is true.

Thus, $P(n)$ implies $P(n+1)$, and since $P(0)$ is true, by induction $P(n)$ is true for all integers $n \geq 0$. ■

5. *Let* $R \subseteq \mathbb{R} \times \mathbb{R}$ *be defined by*

$$(x, y) \in R \text{ iff } x - y \in \mathbb{Z} \text{ or } x + y \in \mathbb{Z}.$$

*Is* $R$ *an equivalence relation? Support your answer with a proof.*

Answer: $R$ is an equivalence relation.

**Proof:** Suppose $R$ is the relation defined above.

**Reflexivity:** Suppose $x \in \mathbb{R}$.

Then $x - x = 0 \in \mathbb{Z}$, so $(x, x) \in R$.

Hence, $x \in \mathbb{R}$ implies $(x, x) \in R$, and so $R$ is reflexive.

**Symmetry:** Suppose $(x, y) \in R$.

Then $x - y \in \mathbb{Z}$ or $x + y \in \mathbb{Z}$.

If $x - y$ is an integer, then $y - x = (-1)(x - y)$ is an integer.

If $x + y$ is an integer, then $y + x = x + y$ is an integer.

Hence, $y - x \in \mathbb{Z}$ or $y + x \in \mathbb{Z}$, and so $(y, x) \in R$.

Thus, $(x, y) \in R$ implies $(y, x) \in R$, and so $R$ is symmetric.

**Transitivity:** Suppose $(x, y) \in R$ and $(y, z) \in R$.

Then $x + y \in \mathbb{Z}$ or $x - y \in \mathbb{Z}$, and $y + z \in \mathbb{Z}$ or $y - z \in \mathbb{Z}$.

Suppose $x + y \in \mathbb{Z}$.

Then $y \pm z \in \mathbb{Z}$, and $x + y - (y \pm z) = x \pm z$, so $x \pm z \in \mathbb{Z}$.

That is, $x + z \in \mathbb{Z}$ or $x - z \in \mathbb{Z}$.

Suppose $x - y \in \mathbb{Z}$.

Then $y \pm z \in \mathbb{Z}$ and $x - y + (y \pm z) = x \pm z$, so $x \pm z \in \mathbb{Z}$

That is $x + z \in \mathbb{Z}$ or $x - z \in \mathbb{Z}$.

Hence, $x + z \in \mathbb{Z}$ or $x - z \in \mathbb{Z}$, so $(x, z) \in R$.

Thus, $(x, y) \in R$ and $(y, z) \in R$ then $(x, y) \in R$, and so $R$ is transitive.

Since $R$ is reflexive, symmetric, and transitive, $R$ is an equivalence relation. ■