Math 301 A - Spring 2014
Final Exam
June 11, 2014

Name: ______________________________ Student ID no. : ____________________

1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
Total	70	

- Complete all seven questions.
- Show all work for full credit.
- You have 110 minutes to complete the exam.
- You may use a non-graphing, scientific calculator. All other electronic devices must be turned off and put away until you have left the room.
- No notes, texts or references of any kind are allowed during the exam.
- You may not share calculators or any other material with other students during the exam.
- You may not speak until you have turned in your exam and left the room.

My signature indicates that I have read and understood the instructions above:

Signature: ____

1. Prove that the product of three consecutive integers is divisible by 60 if the middle integer is a perfect square.

2. Let i be a non-negative integer. Prove that the number

$$8 \cdot 64^i + 25 \cdot 7^i$$

is not prime.

3. Suppose x is the smallest integer greater than 10000 such that

$$3x \equiv 5 \pmod{61} \text{ and } 5x \equiv 3 \pmod{62}.$$

Find x.

4. Find the largest integer that cannot be expressed as a sum of non-negative multiples of $5, 7$ and 13. Prove that this is the largest such integer.

5. Suppose (x, y, z) is a primitive Pythagorean triple, with $z > x$ and $z > y$. Prove that 3 divides exactly one of x and y, and 3 does not divide z. (If you wish to use a result proved in homework, you will need to prove it again here.)

6. Prove that there are no solutions to $5x^2 + 7y^2 = z^2$ with $x, y, z \in \mathbb{Z}$ and $xyz \neq 0$.

7. Define the sequence $\{a_n\}$ by

$$a_0 = 1, a_1 = 1, \text{ and } a_n = a_{n-1} - a_{n-2} \text{ for } n \geq 2.$$

Express the generating function for $\{a_n\}$ as a rational function (i.e., not as a series).

Here is a list of theorems and other facts that you can use without justification during the exam. This is only a partial list! Many minor results may be used without justification. This list is merely a reference of the more powerful and, perhaps, harder to remember ones.

- There are infinitely many primes.
- The transitivity of divisibilty: if a divides b, and b divides c, then a divides c.
- If d divides a and d divides b, then d divides any linear combination of a and b.
- If d and n are positive integers, and d divides n, then $d \leq n$.
- If a prime p divides ab, then p divides a or p divides b.
- The Fundamental Theorem of Arithmetic: all positive integers can be written in a unique way as a product of primes.
- The Euclidean algorithm for finding the gcd of two integers
- Stark, Theorem 2.3: n is a common divisor of a and b iff n divides $\gcd(a, b)$.
- Stark, Theorem 2.6: If $(n, a) = 1$ and $n|ab$, then $n|b$.
- Stark, Theorem 2.13: The n-th root of a positive integer is rational iff it is an integer.
- The result from problem 6 in the week 2 homework (i.e., if $d|n$, then the prime factorization of d consists only of primes from the prime factorization of n, with exponents no greater than the corresponding exponents in the prime factorization of n).
- $\tau(n) = \prod_{p^\alpha || n} (\alpha + 1), \ \sigma(n) = \prod_{p^\alpha || n} \frac{p^{\alpha+1} - 1}{p - 1}, \ \phi(n) = n \prod_{p^\alpha || n} \left(1 - \frac{1}{p}\right)$
- For positive integers a and b, and any prime p,
$$\text{ord}_p ab = \text{ord}_p a + \text{ord}_p b \text{ and } \text{ord}_p(a + b) \leq \min\{\text{ord}_p a, \text{ord}_p b\}$$
- The Chinese Remainder Theorem: Let $m_1, m_2, \ldots, m_k$ be pairwise relatively prime positive integers. Then the system $x \equiv a_1 \pmod{m_1}, x \equiv a_2 \pmod{m_2}, \ldots, x \equiv a_k \pmod{m_k}$ has a unique solution modulo $m_1 m_2 \cdots m_k$.
- Euler's Theorem: Let n be a positive integer. Then $a^{\phi(n)} \equiv 1 \pmod{n}$ if $(a, n) = 1$.
- Frobenius Coin Theorem: Let $a, b > 0$ and $(a, b) = 1$. Then the equation $ax + by = m$ has no non-negative solutions (x, y) if $m = ab - a - b$ and does have solutions if $m > ab - a - b$.
- Primitive Pythagorean Triples: $x^2 + y^2 = z^2$ where $x, y, z \in \mathbb{Z}, x > 0, y > 0, z > 0, (x, y) = 1$ and $2 \mid x$ iff $x = 2ab, y = a^2 - b^2$ and $z = a^2 + b^2$ for some $a, b \in \mathbb{Z}, a > b > 0, (a, b) = 1$, and $a + b \equiv 1 \pmod{2}$.
- Generating functions: The generating function of a sequence $\{a_n\}$ is the function given by the power series $A(x) = \sum_{n=0}^{\infty} a_n x^n$.