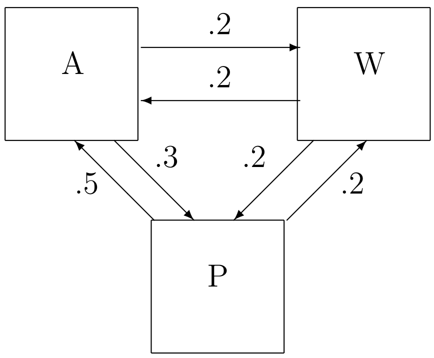
M146 Assignment VII Due Thurs May 18, 2006

Compartment Models, Transfer Matrices

Section 56 (page 639) 10, 11, 12

In compartment models, a substance is transferred between different "compartments". The transfer matrix, T, records the amounts transferred in a unit of time.

Example. The following picture describes the transfer per year of sulfuric acid (SO_2) between A (= atmosphere), W = (water, such as lakes and streams), and P = (plants).



The numbers mean that each year .2 of the SO_2 in A gets transferred to W, .3 of the SO_2 in the A gets transferred to P, etc. (Part of) the transfer matrix for this system is

$$T = \begin{bmatrix} - & .2 & .5 \\ .2 & - & .2 \\ .3 & .2 & - \end{bmatrix}$$

Since $.2 + .3 = .5$ is transferred from A to the other compartments, it must be that .5 is the fraction of SO_2 in A that still remains in A after 1 year. Similarly .6 is the fraction of SO_2 in W that does not get transferred to other compartments, and .5 the fraction of SO_2 in P that does not get transferred to other compartments. The complete transfer matrix is:

$$T = \begin{bmatrix} .5 & .2 & .5 \\ .2 & .6 & .2 \\ .3 & .2 & .3 \end{bmatrix}$$

For each j the entries in column j records the percentages of SO_2 that go to each compartment (including the percentage that remains in compartment j itself). Each column sum is 1.

Suppose that initially the amounts of SO_2 in A, W and P are given by $[x_1, x_2, x_3]$, respectively. Let $[y_1, y_2, y_3]$ denote the amounts at the end of one year. Then, writing X and Y for the column vectors,

$$TX = Y$$

which is the same as saying

$$\begin{array}{cccccc} .5x_1 & +.2x_2 & +.5x_3 & = & y_1 \\ .2x_1 & +.6x_2 & +.2x_3 & = & y_2 \\ .3x_1 & +.2x_2 & +.3x_3 & = & y_3 \end{array}$$

(1a) Let $\text{tot}(X)$ stand for the quantity $x_1 + x_2 + x_3$. If $Y = TX$, show that $\text{tot}(Y) = \text{tot}(X)$.

(1b) Suppose initially: $x_1 = 300\,, x_2 = 0\,, x_3 = 0$. Find the amounts after 1 year, after 2 years, and after 3 years.

(1c) Find the matrix T^2. Show that applying T^2 to the data $[300, 0, 0]$ gives the amount after two years for the previous problem.

A stationary state of the system occurs when the amounts $[x_1, x_2, x_3]$ satisfy:

$$TX = X$$

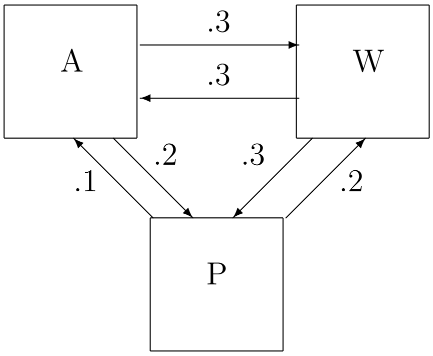
This says that the number 1 is an eigenvalue, and X is an eigenvector for the matrix T.

(1d) Proposition: If M is a square matrix for which every column sum is 1, then 1 must be an eigenvalue for M. Prove this. How does this apply to T?

(1e) For the transfer matrix T, find an eigenvector associated to the eigenvalue 1. Then find an a stationary distribution of 100 units of SO_2.

M146 Sample Quiz #07 for Thursday, May 11, 2006

(1) The following diagram shows the transfer per year of S between three compartments A, W, and P.



Give the transfer matrix for this compartment model.

(1b) Initially there are 1200 tons of substance S in compartment A. Find the steady state distribution of S.

(2) $T = \begin{bmatrix} .8 & 0 & 0 \\ .1 & .6 & .2 \\ .1 & .4 & .8 \end{bmatrix}$ is the transfer matrix for a three compartment model.

Initially there are 1200 tons of substance S in compartment #1. Find the stable distribution.

(3) T is the 5 by 5 matrix for a 5 compartment model. What is the intermpretation of the number $T_{4,3}$.

(4) $T = \begin{bmatrix} .8 & 0 & .1 & .2 \\ .1 & 1 & .2 & .2 \\ .1 & 0 & .6 & .1 \\ 0 & 0 & .1 & .5 \end{bmatrix}$ is the transfer matrix for a four compartment model.

Initially there are 1000 tons of substance S in compartment #1. Find the stable distribution.

(5) $A = \begin{bmatrix} 3 & 1 \\ -3 & 7 \end{bmatrix}$ Find two solutions to $Av = \lambda v$ where v is a vector and λ is a scalar.

(6) Suppose T is a square matrix for which every column sum is 1. Explain why there is a non-zero vector X with $TX = X$.