M146 Final Monday, June 5, 2006 **SOLUTIONS**

(1) Substance S is taken into the stomach, is absorbed by the blood, and removed from the blood by the kidney. The half-life of S in the stomach is 1 hour. The half-life of S in the blood is 5 hours. Initially there are 100 mg of S in the stomach, none in the blood.

(1a) Calculate the amount of S in the blood for all times t.

$$r_1 = \ln 2 = .693 \quad r_2 = \frac{\ln 2}{5} = .1386 \quad \begin{matrix} x_1' \\ x_2' \end{matrix} = \begin{matrix} -.693x_1 & 0 \\ .693x_1 & -.1386x_2 \end{matrix}$$

That is, $x' = Ax$, where $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $A = \begin{bmatrix} -.693 & 0 \\ .693 & -.1386 \end{bmatrix}$

$A - \lambda I = \begin{bmatrix} -.693-\lambda & 0 \\ .693 & -.1386-\lambda \end{bmatrix}$, the eigenvalues are $\lambda_1 = -.693$, $\lambda_2 = -.1386$

(1) $\lambda_1 = -.693$, $A - \lambda I = \begin{bmatrix} 0 & 0 \\ .693 & .5545 \end{bmatrix} \to \begin{bmatrix} 0 & 0 \\ 1.25 & 1 \end{bmatrix}$, $v_1 = \begin{bmatrix} .8 \\ -1 \end{bmatrix}$,

(2) $\lambda_2 = -.1386$, $A - \lambda I = \begin{bmatrix} -.5545 & 0 \\ .693 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$,

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = C_1 \begin{bmatrix} .8 \\ -1 \end{bmatrix} e^{-.693t} + C_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-.1386t}$$

$$\begin{bmatrix} 100 \\ 0 \end{bmatrix} = C_1 \begin{bmatrix} .8 \\ -1 \end{bmatrix} + C_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_1 = 125, \quad C_2 = 125$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 100 \\ -125 \end{bmatrix} e^{-.693t} + \begin{bmatrix} 0 \\ 125 \end{bmatrix} e^{-.1386t}$$

(1b) What is the maximum amount of S in the blood, and at what time does this maximum occur?

$x_2 = -125 \cdot e^{-.693t} + 125 \cdot e^{-.1386t}$,

x_2 has a maximum when $t = 2.9$, and $x_2 = 66.9$

(2a) Find the steady state values for the differential equation $y' = 2\,y(y-5)(100-y)$. For each steady state value determine whether it is stable, or unstable..

$y = 0$ is a stable critical value.

$y = 5$ is an unstable critical value.

$y = 100$ is a stable critical value.

(2b) If $y = f(t)$ solves **(2a)** with $f(0) = 4$, $\lim_{t\to\infty} f(t) = 0$.

(2c) If $y = f(t)$ solves **(2a)** with $f(0) = 6$, $\lim_{t\to\infty} f(t) = 100$.

(2d) In the absence of predators, a population of rabbits grows exponentially and would double in 7 months. A pack of foxes eat rabbits at a rate of 10 rabbits per month. Initially there are 120 rabbits. Calculate the number of rabbits for all times t.

$$r = \frac{\ln 2}{7} = 0.099,$$

$$\frac{dy}{dt} = 0.099y - 10 = .099(y - 101)$$

$$\int \frac{dy}{y-101} = \int .099\,dt \quad \text{(separating} \quad \text{variables)}$$

$$\ln(y - 101) = e^{.099t} + C$$

$$y - 101 = Ae^{.099t} \qquad 120 - 101 = A; \quad 19 = A$$

$$y = 101 + 19e^{.099t}$$

(2e) Answer is (B) The population of rabbits increases without limit. They double when

$$120 = 101 + 19e^{.099t}, \quad 0.099t = \ln\left(\frac{120}{19}\right), \quad t \approx 20$$

(3) $A = \begin{bmatrix} 1 & 3 & 4 & -1 \\ 2 & 6 & 9 & 3 \\ 1 & 3 & 5 & 4 \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$.

(3a) What is the rank of A? What is the nullity of A?

$A \to \begin{bmatrix} 1 & 3 & 0 & -21 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, so the rank is 2, the nullity is 4-2 = 2.

(3b) The solutions of $Ax = 0$ are $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 21 \\ 0 \\ -5 \\ 1 \end{bmatrix} \cdot c_1 + \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} \cdot c_2$

(3c) $\left[\begin{array}{cccc|c} 1 & 3 & 4 & -1 & 1 \\ 2 & 6 & 9 & 3 & 4 \\ 1 & 3 & 5 & 4 & 3 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 3 & 0 & -21 & -7 \\ 0 & 0 & 1 & -5 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$,

The solutions of $Ax = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$ are $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 21 \\ 0 \\ -5 \\ 1 \end{bmatrix} \cdot c_1 + \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} \cdot c_2 + \begin{bmatrix} -7 \\ 0 \\ 2 \\ 0 \end{bmatrix}$

(4) The table shows the length-weight relation for a species of Halibut. Use the method of least squares to find a power function $W = \alpha L^\beta$ which best fits the data.

L	0.4	1.0	2.5
W	1	10	100

Take ln of the L values and ln of the W values to get: $\ln(W) = \ln(\alpha) + \beta \ln(L)$, an equation of the form $z = b + mx$.

$x = \ln L$	-.9163	0	.9163
$z = \ln W$	0	2.3	4.6

We want the line $z = b + mx$ which best fits this data.

To find b and m so that $\begin{array}{rcrcr} b & - & 0.9163m & = & 0 \\ b & + & 0 & = & 2.30 \\ b & + & 0.9163m & = & 4.6 \end{array}$, an overdetermined system

$$A^T \cdot A = \begin{bmatrix} 1 & 1 & 1 \\ -.693 & 0 & .693 \end{bmatrix} \cdot \begin{bmatrix} 1 & -.9163 \\ 1 & 0 \\ 1 & .9163 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 1.679 \end{bmatrix}, \quad \text{and}$$

$$A^T \cdot c = \begin{bmatrix} 1 & 1 & 1 \\ -.9163 & 0 & .9163 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 2.3 \\ 4.6 \end{bmatrix} = \begin{bmatrix} 6.91 \\ 4.22 \end{bmatrix}.$$

$$\begin{bmatrix} 3 & 0 \\ 0 & 1.679 \end{bmatrix} \cdot \begin{bmatrix} b \\ m \end{bmatrix} = \begin{bmatrix} 6.91 \\ 4.22 \end{bmatrix}$$

The solution is $\begin{bmatrix} b \\ m \end{bmatrix} = \begin{bmatrix} 2.3 \\ 2.5 \end{bmatrix}$, The "best" line is $z = 2.3 + 2.5x$

The power law is $W = e^{2.3} L^{2.46} = 10L^{2.5}$

(5) A population is divided into newborns, yearlings and adults. Each adult produces 2.2 newborns per year. Each year 55 % of the newborns survive to be yearlings, 70 % of yearlings survive to become adults and 40 % of the adults survive to the next year. After many generations the percentages of newborns, yearlings and adults in the population stabilizes. Find the stable percentages of newborns, yearlings and adults. (The largest eigenvalue for the Leslie matrix is $\lambda = 1.1$)

The Leslie matrix for this population is $L = \begin{bmatrix} 0 & 0 & 2.2 \\ 0.55 & 0 & 0 \\ 0 & 0.7 & 0.4 \end{bmatrix}$.

$$L - \lambda_1 I = \begin{bmatrix} -1.1 & 0 & 2.2 \\ 0.55 & -1.1 & 0 \\ 0 & 0.7 & -0.7 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{Solution is} \quad \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.25 \\ 0.25 \end{bmatrix}$$

(6) A certain ecosystem consists of A (= atmosphere), W (= water) and P = (plants), each of which contains sulfuric acid (SO_2). No SO_2 enters or leaves the ecosystem.

20 % of the SO_2 in the Air is transferred to the Water;
10 % of the SO_2 in the Air is transferred to the Plants;

30 % of the SO_2 in the Water is transferred to the Air;
10 % of the SO_2 in the Water is transferred to the Plants;

30 % of the SO_2 in the Plants is transferred to the Air;
20 % of the SO_2 in the Plants is transferred to the Water;

Find the stationary state distribution of 1200 lbs of SO_2.

The complete transfer matrix is: $T = \begin{bmatrix} .7 & .3 & .3 \\ .2 & .6 & .2 \\ .1 & .1 & .5 \end{bmatrix}$

$$T - I = \begin{bmatrix} -.3 & .3 & .3 \\ .2 & -.4 & .2 \\ .1 & .1 & -.5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{The eigenvector is} \quad \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \cdot a$$

We must have $3a + 2a + a = 1200$, $a = 200$, so the solution is $\begin{bmatrix} 600 \\ 400 \\ 200 \end{bmatrix}$

(7a) $u = 3 + 4i, \quad v = 3 + i, \quad w = \dfrac{u}{v},$ Express w in rectangular from. Plot u, v and w in the cpmplex plane.

$$w = \frac{3+4i}{3+i} = \frac{(3+4i)(3-i)}{(3+i)(3-i)} = \frac{13+9i}{10} = 1.3 = .9i,$$

(7b) $a = 2 + i \quad b = -4 + 2i, \quad c = a \cdot b$ express a, b and c in polar form. Plot a, b and c in the cpmplex plane.

$$a = \sqrt{5}e^{0.46i} \quad b = \sqrt{20}e^{(\pi - 0.46i)} \quad a \cdot b = \sqrt{100}e^{(0.46 + \pi - 0.46i)} = 10e^{\pi i}$$

(7c) Find all complex numbers z which satisfy $z^3 = -8i$. Plot these in the complex plane.

$z^3 = w = 8e^{\frac{3\pi}{2}} = 8e^{\frac{7\pi}{2}} = 8e^{\frac{11\pi}{2}},$ The 3 cube roots of w are

$z_1 = 2e^{\frac{\pi}{2}} \quad z_2 = 2e^{\frac{7\pi}{6}} \quad z_3 = 2e^{\frac{11\pi}{6}}$