M308 Sample Problems for Final (A)

(1a) Let $A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & 7 \\ 3 & -5 & -7 \end{bmatrix}$ and $b = \begin{bmatrix} 2 \\ 8 \\ -8 \end{bmatrix}$

Form the augmented matrix, use ERO's to transfrom to RREF, and find all solutions to $Ax = b$.

(2) $A = \begin{bmatrix} 1 & 2 & 1 & -1 & 1 \\ 2 & 4 & 1 & -4 & -3 \\ -1 & -2 & 1 & 5 & 5 \end{bmatrix}$ $\quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$ $\quad d = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$

(2a) Find all solutions of $Ax = 0$.

(2b) Find all solutions of $Ax = d$.

(2c) Is there a vector b for which $Ax = b$ has no solution x? If so, find one.

(3a) Given a set $\{v_1, \ldots, v_k\}$ of vectors in R^n, and another vector w, how can you decide if w is a linear combination of $v_1, \ldots v_k$?

Answer: See **(3b)** and **(3c)**

(3b) In particular, let $v_1 = (1, 3, 2)$, $v_2 = (2, 1, -1)$, $w = (3, -1, -4)$. Is w a linear combination of v_1 and v_2?

Solution: Let A be the matrix whose columns are v_1, v_2. Then

$$[A|w] = \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 3 & 1 & -1 \\ 2 & -1 & 4 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & -5 & -10 \\ 0 & -5 & -10 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$$

The third row of 0's shows that there is a solution to $v_1x_1 + v_2x_2 = w$, namely $x_1 = -1$ and $x_2 = 2$, so $\quad -v_1 + 2v_2 = w$,

(3c) $u = (2, 6, 2)$. Is u a linear combination of v_1 and v_2?

$$[A|u] = \left[\begin{array}{cc|c} 1 & 2 & 2 \\ 3 & 1 & 6 \\ 2 & -1 & 2 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & -5 & 0 \\ 0 & -5 & -2 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & -5 & 0 \\ 0 & 0 & -2 \end{array}\right]$$

The last row is inconsistent, so u is not a linear combination of v_1 and v_2.

(4a) What is the definition that a set of vectors $v_1, v_2, \ldots v_k$ are <u>linearly dependent</u>?

Answer: There are constants $c_1, \ldots, c_k$, not all 0, so that $v_1c_1 + v_2c_2 + \ldots v_kc_k = 0$

(4b) What is the definition that a set of vectors $v_1, v_2, \ldots v_k$ are <u>linearly independent</u>?

Answer: If $v_1c_1 + v_2c_2 + \ldots v_kc_k = 0,$ then all c_i must be 0.

(5a) If $\{v_1, \ldots, v_k\}$ is a set of vectors in R^n, how can you tell if they are linearly dependent or linearly independent?

(5b) In particular, are $v_1 = [1, 3, 2]$, $v_2 = [1, -1, 2]$, $v_3 = [1, 7, 2]$ linearly independent?

Solution: Let A be the matrix whose columns are $v_1,\ v_2, v_3$. $A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & -1 & 7 \\ 2 & 2 & 2 \end{bmatrix}$ Then

$A \rightsquigarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, so the vectors v_1, v_2 and v_3 are linearly dependent.

(5c) In particular, are $w_1 = [1, 3, 2]$, $w_2 = [1, -1, 2]$, $w_3 = [1, 7, 2]$ linearly independent?

Solution: Let A be the matrix whose columns are $w_1,\ w_2, w_3$. $A = \begin{bmatrix} 1 & 1 & 2 \\ 3 & -1 & 6 \\ 2 & 2 & 2 \end{bmatrix}$ Then

$A \rightsquigarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, so the vectors w_1, w_2 and w_3 are linearly in dependent.

(6a) What is the definition that a set of vectors $v_1, v_2, \ldots v_m$ <u>span</u> a vector space W?

Answer: Every vector in V is a linear combination of $v_1, v_2, \ldots v_m$.

(6b) Given a set $\{v_1, \ldots, v_k\}$ of vectors in R^n, and another vector w, how can you tell if $w \in \text{span}\ (v_1, \ldots v_k)$?

(6c) $v_1 = (1, 3, 2)$, $v_2 = (2, 1, -1)$, and $w = (3, -1, -4)$. Is $w \in \text{span}\ (v_1, v_2)$?

(7a) What is the definition that a set of vectors $v_1, v_2, \ldots v_m$ are a <u>basis</u> for W?

(7b) Suppose $v_1, \ldots, v_k$ span a vector space V. How do you find a basis for V? How can you find a subset of $v_1, \ldots, v_k$ that is basis for V?

(7c) V is spanned by $v_1 = (1, 1, 2)$, $v_2 = (2, 1, -1)$, $v_3 = (3, 1, -4)$. Find a basis for V.

(8a) What is the definition of the <u>nullspace</u> of a matrix A?

(8b) What is the definition of the rowspace of a matrix A?

(8c) What is the definition of the columnspace of a matrix A?

(8d) A is a 9 by 13 matrix; A reduces to a matrix in RREF with 2 zero rows. What are the dimensions of the nullspace, rowspace and columnspace of A?

(9a) $A = \begin{bmatrix} 1 & 2 & 3 & 1 & 6 & 5 \\ 1 & 2 & 3 & 2 & 9 & 7 \\ 2 & 4 & 6 & 1 & 9 & 8 \end{bmatrix}$ Find a basis for the rowspace of A

Solution: A reduces to $R = \begin{bmatrix} 1 & 2 & 3 & 0 & 3 & 3 \\ 0 & 0 & 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

(9a) A basis for ROW_A consists of the two non-zero rows of R.
(9b) Find a basis for the nullspace of A.

A basis for N_A is $v_1 = \begin{bmatrix} -3 \\ 0 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} -3 \\ 0 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix}$, $v_3 = \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $v_4 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$,

(9c) Find a basis for the columnspace of A.

(9d) (Harder) Find a subset of the columns which form basis for the columnspace of A.

Solution: The first and fourth columns of A are a basis for $C_!$. Why?

(10) Find the inverse of the matrix $A = \begin{bmatrix} 1 & 2 & -4 \\ -1 & -1 & 5 \\ 2 & 7 & -4 \end{bmatrix}$

(11) A 3×3 matrix A reduces to the identity by the following row operations:

1) Add $2R_1$ to R_2
2) Add $-3R_1$ to R_3
3) Multiply R_3 by $1/2$
4) Add $-R_3$ to R_2
5) Add $-3R_2$ to R_1

(11a) What is A^{-1}?

(11b) Find the solution to $Ax = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$.

(12) Let $v = (2, -1)$ and $w = (7, 4)$. Find a number c so that $w - cv$ is perpendicular to v.

(13) Let $v_1 = [1, 0, 1]$, $v_2 = [2, 2, 1]$ and $w = [2, 3, 5]$. Let V be the plane spanned by v_1, v_2. Find a vector v_3 in V and a vector $y \perp V$ so that $w = v_3 + y$.

Solution We want $v_3 = v_1 c_1 + v_2 c_2$ so that $v_1 c_1 + v_2 c_2 - w$ is perpendicular to V.

Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}$. We must solve $A^T\big(A\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} - w\big) = 0$. That is

$A^T A \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = A^T w.$

$\begin{bmatrix} 2 & 3 \\ 3 & 9 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 15 \end{bmatrix}$. The solution is $c_1 = 2$, $c_2 = 1$, so $v_3 = [4, 2, 3]$ and $y = [-2, 1, 2]$.

(13) What 2 by 2 matrix A rotates every vector in the plane counterclockwise by $45°$?

Solution: $A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$ and $A\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$, so $A = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$

(14a) Let A be an $n \times n$ matrix. Give two different ways to calculate $\det A$?

Answer: (i) Use ERO's $E_1, E_2, \ldots, E_s$ to reduce A to an upper triangular matrix, keeping track of the multipliers $m_1 \cdot m_2 \ldots m_s$. Then $\det A = m_1^{-1} \cdot m_2^{-1} \ldots m_s^{-1}$.

(ii) Pick any one row or column, and expand by cofactors. For example, using the first row,

$\det A = a_{11} \det(M_{11}) - a_{12} \det(M_{12}) + \ldots + (-1)^{1+n} a_{1n} \det(M_{1,n}$

(14b) Calculate $\det A$ for $A = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 4 & -2 & 5 \\ 2 & 9 & 1 & 9 \\ 1 & 7 & 4 & 5 \end{bmatrix}$

(15) $A = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$. Find two eigenvalues λ_1, λ_2 and corresponding eigenvectors v_1, v_2.

(16) Let T be the linear transformation from R^3 to R^2 given by the formula:

$T[x, y, z] = [4x + y + z, -x + 2y - 3z]$. Find the matrix of T with respect to the standard basis.

Answer $A = \begin{bmatrix} 4 & 1 & 1 \\ -1 & 2 & -3 \end{bmatrix}$.

(Complex) Compute each of the following (Answer in the form $x + iy$):

(1a) $(\sqrt{2} - i) - i(1 - \sqrt{2}i)$ Answer $-2i$

(1b) $\dfrac{1 + 2i}{3 - 4i} + \dfrac{2 - i}{5i}$ Answer $-\dfrac{2}{5}$

(1c) $\dfrac{5}{(1 - i)(2 - i)(3 - i)}$ Answer $\dfrac{1}{2}i$

(1d) $(1 - i)^4$ Answer -4