M308 Sample for Test #2, Nov 19, 2007 with some **SOLUTIONS**

(4) Suppose V and W are vector spaces. What is the definition of a linear transformation T from V to W.

Answer: T is a function from V to W with the property that $T(au+bv) = aT(u)+bT(v)$ for all scalars a, b and all vectors u and v in V.

Alternative answer: T is a function from V to W with the property that

(1) $T(u+v) = T(u) + T(v)$ for all vectors u, v in V.

(2) $T(cu) = cT(u)$, for c a scalar and u any vector in V.

(5) Let T be the linear transformation of the plane to itself for which $T(1,2) = (1,4)$ and $T(3,7) = (5,2)$. Find $T(4,7)$.

Solution: First express $\begin{bmatrix} 4 \\ 7 \end{bmatrix}$ as a linear combination: $\begin{bmatrix} 4 \\ 7 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot x_1 + \begin{bmatrix} 3 \\ 7 \end{bmatrix} \cdot x_2.$

Solving $\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \end{bmatrix}.$ we get $x_1 = 7$, $x_2 = -1$. Then

$$T \begin{bmatrix} 4 \\ 7 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \cdot 7 + \begin{bmatrix} 5 \\ 2 \end{bmatrix} \cdot (-1) = \begin{bmatrix} 2 \\ 26 \end{bmatrix}.$$

(6) Let T be the linear transformation from R^3 to R^3 with $T(x,y,z) = (x-y, y-z, z-x)$. Find the matrix which represents T with respect to the standard basis.
Solution:

$$T(e_1) = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad T(e_2) = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad T(e_3) = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \quad \text{so} \quad A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}.$$

(7) Let T be the linear transformation from R^3 to R^2 with $T(e_1) = 2e_1 - e_2$, $T(e_1 + e_2) = 4e_1 + 3e_2$, $T(e_1 + e_2 + e_3) = 7e_1 + 4e_2$, Find the matrix which represents T with respect to the standard basis.

Solution: $T(e_1) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$,

$$T(e_2) = T(e_1 + e_2) - T(e_1) = \begin{bmatrix} 4 \\ 3 \end{bmatrix} - \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix},$$

$$T(e_3) = T(e_1 + e_2 + e_3) - T(e_1 + e_2) = \begin{bmatrix} 7 \\ 4 \end{bmatrix} - \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \text{ so } A = \begin{bmatrix} 2 & 2 & 3 \\ -1 & 4 & 1 \end{bmatrix}.$$

(8) Find the eigenvalues and eigenvectors of the matrix $B = \begin{bmatrix} 4 & -2 \\ 3 & -1 \end{bmatrix}$

Solution: $\det \begin{bmatrix} 4 - \lambda & -2 \\ 3 & -1 - \lambda \end{bmatrix} = \lambda^2 - 3\lambda + 2 = (\lambda - 1)(\lambda - 2)$

The eigenvalues are $\lambda_1 = 1$, and $\lambda_2 = 2$.

For $\lambda_1 = 1$, $B - \lambda_1 \cdot I = \begin{bmatrix} 3 & -2 \\ 3 & -2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -\frac{2}{3} \\ 0 & 0 \end{bmatrix}$, the eigenvector is $v_1 = \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}$,

For $\lambda_2 = 1$, $B - \lambda_2 \cdot I = \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$, the eigenvector is $v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$,

(9) Calculate $\det A$ for $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 2 \\ 2 & 9 & 5 \end{bmatrix}$. $\det(A) = 7 - 12 + 8 = 3$ (Not on Quiz Feb 23)

(10) Calculate the determinant of the matrix $A = \begin{bmatrix} 4 - \lambda & 2 \\ 1 & 3 - \lambda \end{bmatrix}$

Solution: $\det \begin{bmatrix} 4 - \lambda & 2 \\ 1 & 3 - \lambda \end{bmatrix} = \lambda^2 - 7\lambda + 10 = (\lambda - 2)(\lambda - 5)$