M308 Test #2 SOLUTIONS

(1) What is the <u>definition</u> of a linear transformation T from a vector space V to another vector space W

Answer: For any scalars a and b and vectors u and w in V, $T(au+bv) = aT(u)+bT(v)$.

(2) Let T be a linear transformation from R^5 to R^3 of rank 2. What is the nullity of T?

Solution The number of variables is 5, the rank is 2, so the nullity is $5-2=3$.

(3) Let T be a linear transformation from R^3 to R^4, and the only solution to $Tx=0$ is $x=0$. What is the rank of T?

Solution The number of variables is 3, the nullity is 0, so the rank is $3-0=3$.

(4) Measurements are made at three locations: $f(-1)=-2$, $f(1)=2$ and $f(2)=3$. Find the line $y=b+mx$ which gives the best least squares fit to this data.

Solution We want the "best" solution to
$$\begin{array}{rcrcr} b & - & m & = & -2 \\ b & + & m & = & 2 \\ b & + & 2m & = & 3 \end{array}$$

$$A^T\cdot A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix}\cdot\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix} \quad A^T\cdot b = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix}\cdot\begin{bmatrix} -2 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 10 \end{bmatrix}$$

The solution to $\begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix}\cdot\begin{bmatrix} b \\ m \end{bmatrix} = \begin{bmatrix} 3 \\ 10 \end{bmatrix}$ is $\begin{bmatrix} b \\ m \end{bmatrix} = \begin{bmatrix} -\frac{1}{7} \\ \frac{12}{7} \end{bmatrix}$

The "best" line is $y = -\dfrac{1}{7} + \dfrac{12}{7}$.

(5) Let T be the linear transformation from R^2 to R^3 with $T(x,y) = (x+3y, 2x-y, y-x)$.

Find the matrix which represents T with respect to the standard basis.

Solution $T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \quad T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}; \quad \text{so } A = \begin{bmatrix} 1 & 3 \\ 2 & -1 \\ -1 & 1 \end{bmatrix}$

(6) $v = [2,1,2], \quad w = [1,6,5]$. Find a scalar c and a vector y with $w = cv + y$, and $y \perp v$.

Solution $v \cdot w = v \cdot (cv + y) = v \cdot v + v \cdot y = c \cdot v \cdot v + 0$

$[2,1,2] \cdot [1,6,4] = 18 = [2,1,2] \cdot [2.1.2]c = 9c$

$c = 2, \quad y = w - 2v = 1,5,6] - [4,2,4] = [-3,4,1]$

(7) Find the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$

Solution $\det \begin{bmatrix} 4-t & 2 \\ 1 & 3-t \end{bmatrix} = t^2 - 7t + 10 = (t-2)(t-5). \quad t_1 = 2, \quad t_2 = 5$

For $t_1 = 2$, $\begin{bmatrix} 4-2 & 2 \\ 1 & 3-2 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}; \quad v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

For $t_2 = 5$, $\begin{bmatrix} 4-5 & 2 \\ 1 & 3-5 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix}; \quad v_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$